MA342A (Harmonic Analysis 1) Tutorial sheet 1
[October 4, 2017]

Name: Solutions

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1. Express $f(x) = \cos(4\pi x) + 7\cos(8\pi x)$ in terms of (*i.e.* as a linear combination of) the complex exponentials $\phi_n(x) = e^{2\pi i n x}$ ($n \in \mathbb{Z}$). Deduce the complex Fourier coefficients $\hat{f}(n)$.

   Solution: We know $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and so

$$\begin{aligned}
\cos(4\pi x) &= \frac{1}{2}(e^{4\pi i x} + e^{-4\pi i x}) \\
&= \frac{1}{2}(\phi_2(x) + \phi_{-2}(x)) \\
\cos(8\pi x) &= \frac{1}{2}(e^{8\pi i x} + e^{-8\pi i x}) \\
&= \frac{1}{2}(\phi_4(x) + \phi_{-4}(x))
\end{aligned}$$

   and so

$$f(x) = \frac{1}{2}\phi_2(x) + \frac{1}{2}\phi_{-2}(x) + \frac{7}{2}\phi_4(x) + \frac{7}{2}\phi_{-4}(x).$$

   Since the ϕ_n are orthormal, we deduce

$$\hat{f}(n) = \begin{cases} \frac{1}{2} & \text{if } n = \pm 2 \\ \frac{7}{2} & \text{if } n = \pm 4 \\ 0 & \text{otherwise} \end{cases}$$

2. Let $f\colon [0,1] \to \mathbb{R}$ be integrable. Show that

$$\hat{f}(-n) = \overline{\hat{f}(n)} \qquad (n \in \mathbb{Z}).$$

   Solution: We know

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx$$

and so

$$\begin{aligned}
\hat{f}(-n) &= \int_0^1 f(x)e^{2\pi inx}\,dx \\
&= \int_0^1 f(x)\overline{e^{-2\pi inx}}\,dx \\
&= \int_0^1 \overline{f(x)e^{-2\pi inx}}\,dx \\
&\quad \text{(as } f(x) \in \mathbb{R}\text{)} \\
&= \overline{\int_0^1 f(x)e^{-2\pi inx}\,dx} \\
&\quad \text{(easy to justify from the definition of integrals of } \mathbb{C}\text{-valued functions)} \\
&= \overline{\hat{f}(n)}
\end{aligned}$$

Aside: This means that for $n = 0$ we get $\hat{f}(0) = \overline{\hat{f}(0)}$ and that means $\hat{f}(0)$ is real (if f is $\mathbb{R}$-valued). It also means that we can combine terms in the Fourier series for $n \neq 0$ to get

$$\hat{f}(n)e^{2\pi inx} + \hat{f}(-n)e^{-2\pi inx} = A_n \cos(2\pi nx) + B_n \sin(2\pi nx)$$

with $A_n = 2\Re\hat{f}(n)$ and $B_n = -2\Im\hat{f}(n)$ (real).

3. Let $f \in CP[0,1]$ be given by $f(x) = x - x^2$. Find the Fourier coefficients $\hat{f}(n)$ ($n \in \mathbb{Z}$).

   Solution: We want to compute $\hat{f}(n) = \int_0^1 (x - x^2)e^{-2\pi inx}\,dx$. Using integration by parts

$$\begin{aligned}
\int_0^1 xe^{-2\pi inx}\,dx &= \int_0^1 xd\left(\frac{1}{-2\pi in}e^{-2\pi inx}\right) \\
&= \left[-\frac{x}{2\pi in}e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in} + 0 + \frac{1}{2\pi in}\left[\frac{1}{-2\pi in}e^{-2\pi inx}\right]_0^1 \\
&= -\frac{1}{2\pi in} = \frac{i}{2\pi n}
\end{aligned}$$

if $n \neq 0$.

$$\begin{aligned}
\int_0^1 x^2 e^{-2\pi i n x}\,dx &= \int_0^1 x^2 d\left(\frac{1}{-2\pi i n} e^{-2\pi i n x}\right) \\
&= \left[-\frac{x^2}{2\pi i n} e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n x} 2x\,dx \\
&= \frac{i}{2\pi n} + \frac{1}{\pi i n}\int_0^1 x e^{-2\pi i n x}\,dx \\
&= \frac{i}{2\pi n} + \frac{1}{\pi i n}\frac{i}{2\pi n} \\
&= \frac{1}{2\pi^2 n^2} + \frac{i}{2\pi n}
\end{aligned}$$

(again if $n \neq 0$). So, if $n \neq 0$,

$$\hat{f}(n) = \frac{i}{2\pi n} - \left(\frac{1}{2\pi^2 n^2} + \frac{i}{2\pi n}\right) = -\frac{1}{2\pi^2 n^2}$$

For $n = 0$ we have

$$\hat{f}(0) = \int_0^1 x - x^2\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}.$$

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