Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

MA342A-sampl

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS & SS Mathematics **Trinity Term 2016**
JS & SS TSM

MA342A — Harmonic analysis I

? ? 14.00 — 16.00

Professor R. M. Timoney

Instructions to Candidates:

Credit will be given for the best THREE questions answered.

All questions have equal weight.

'Formulae & tables' are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (8 points) Define the space $\mathcal{L}^1[0,1]$ and the usual semi-norm $\|\cdot\|_1$ on it. Explain the definition of a seminorm and prove that $\|\cdot\|_1$ is indeed a seminorm on $\mathcal{L}^1[0,1]$.

   (b) (4 points) Let $f \in \mathcal{L}^1[0,1]$ be given by $f(x) = x$ for $0 \leq x \leq 1/3$ and $f(x) = 0$ otherwise. Calculate the Fourier series for f.

   (c) (8 points) Define what a character χ of the real line $\mathbb{R}$ is, and explain how the dual group $\hat{\mathbb{R}}$ is defined. Outline in detail a proof that $\hat{\mathbb{R}}$ can be identified with $\mathbb{R}$.

2. (a) (8 points) Define the terms topological abelian group G, LCA group G, dual group $\hat{G}$. If G is a a finite abelian group, explain how it is usually considered to be an LCA group.

   (b) (6 points) Outline the definition of a Haar measure and the Fourier transform $\hat{F}$ for a function F on an LCA group G.

   (c) (6 points) State Plancherel's theorem for an LCA group G and explain how it relates to Parseval's identity for Fourier series in the case when $G = \mathbb{T}$.

3. (a) (6 points) If $f : \mathbb{R} \to \mathbb{C}$ is periodic with period 1 and has a continuous derivative $h = f'$, show that

$$\hat{h}(n) = 2\pi i n \hat{f}(n)$$

   and use Bessel's inequality to deduce that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

   (b) (6 points) Show that for f as in the previous part, the Fourier series for f is uniformly convergent (to some $g \in CP[0,1]$).

   (c) (8 points) State Fejér's theorem in a general setting and outline how it can be used to show that $g = f$ in the previous part.

4. (a) (6 points) Define the convolution $F * H$ of two functions $F, H \in L^1(\mathbb{T})$ and show how Fubini's theorem can be used to show that the definition makes sense. Show also that $F * H \in L^1(\mathbb{T})$.

   (b) (6 points) If $\chi_n(\zeta) = \zeta^n$ ($n \in \mathbb{Z}$, $\zeta \in \mathbb{T}$), compute $\chi_n * F$ in terms of $\hat{F}$ for $F \in L^1(\mathbb{T})$.

   (c) (8 points) Outline a proof that if $F \in L^1(\mathbb{T})$ and $\phi \in C(\mathbb{T})$, then $\phi * F$ agrees (almost everywhere) with the $L^1(\mathbb{T})$-valued integral

$$\int_0^1 \phi(e^{2\pi i x}) F_{e^{2\pi i x}}\, dx,$$

   where $F_\zeta(\eta) = F(\zeta^{-1}\eta)$ for $\zeta, \eta \in \mathbb{T}$.