MA3422 (Functional Analysis 2) Tutorial sheet 8
[March 30, 2017]

Name: Solutions

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1. Let E be a normed space. Prove that $\dim E$ infinite implies $\dim E^*$ is infinite.

   Solution: If $\dim E^* < \infty$, then its dual $(E^*)^* = E^{**}$ would have the same finite dimension. Since there is a canonical (linear isometric) embedding $J \colon E \to E^{**}$, we'd have to have $\dim E < \infty$ also.

2. Let E be a normed space and suppose that $(x_n)_{n=1}^\infty$ is a sequence in E such that $\lim_{n\to\infty} \alpha(x_n) = 0$ for each $\alpha \in E^*$. Show that $(x_n)_{n=1}^\infty$ converges to $0 \in E$ in the weak topology of E. [Hint: If $U \subset E$ is open in the weak topology and $0 \in E$, then there is a seminorm $q \colon E \to [0, \infty)$ continuous for the weak topology such that $B_q(0, 1) \subset U$. Also there is some m, some $\alpha_1, \alpha_2, \ldots, \alpha_m \in E^*$ and some constants $C_1, C_2, \ldots, C_m > 0$ with $q(x) \le C_1|\alpha_1(x)| + \cdots + C_m|\alpha_m(x)|$ for $x \in E$.]

   Solution: We aim to prove that if $U \subset E$ is open in the weak topology then there is N such that $n \ge N \Rightarrow x_n \in U$.

   Fix such a U.

   As in the hint, there is a seminorm $q \colon E \to [0, \infty)$ continuous for the weak topology such that $B_q(0, 1) \subset U$. And we have $q(x) \le C_1|\alpha_1(x)| + \cdots + C_m|\alpha_m(x)|$ holding (for all $x \in E$ and some $\alpha_1, \alpha_2, \ldots, \alpha_m \in E^*$, some constants $C_1, C_2, \ldots, C_m > 0$).

   By the assumption, $\lim_{n\to\infty} \alpha_j(x_n) = 0$ (for $1 \le j \le m$) and so there is N_j such that

   $$n \ge N_j \Rightarrow |\alpha_j(x_n)| < \frac{1}{mC_j}$$

   Now for $N = \max_{1\le j\le m} N_j$, we have

   $$n \ge N \Rightarrow q(x_n) \le \sum_{j=1}^m C_j|\alpha_j(x_n)| \le \sum_{j=1}^m C_j \frac{1}{mC_j} = 1 \Rightarrow x_n \in B_q(0, 1) \subset U$$

3. Let $E = c_0$ and $\bar{B}_E = \{x \in E : \|x\| \le 1\}$ (where we use the usual norm $\|\cdot\|_\infty$ on c_0). Let $(x_n)_{n=1}^\infty$ be a sequence in $\bar{B}_E$ and write each x_n as

   $$x_n = (x_{n,j})_{j=1}^\infty = (x_{n,1}, x_{n,2} \ldots)$$

   for scalars $x_{n,j} \in \mathbb{K}$. Show that if $\lim_{n\to\infty} x_{n,j} = 0$ for each j, then the sequence $(x_n)_{n=1}^\infty$ converges to $0 \in c_0$ in the weak topology. [Hint: Recall c_0^* can be identified with ℓ^1. Use the previous question.]

Solution: From the previous question it is enough to show that $\lim_{n\to\infty} \alpha(x_n) = 0$ for each $\alpha \in c_0^*$.

If $\alpha \in c_0^*$, we know (hint) that there is $y = (y_j)_{j=1}^\infty \in \ell^1$ so that

$$\alpha\left((x_j)_{j=1}^\infty\right) = \sum_{j=1}^\infty x_j y_j \qquad ((x_j)_{j=1}^\infty \in c_0)$$

Now if $\varepsilon > 0$ is fixed we can choose J so large that

$$\sum_{j=J+1}^\infty |y_j| < \frac{\varepsilon}{2}$$

(since $\sum_{j=1}^\infty |y_j| < \infty$). Then

$$\begin{aligned}
|\alpha(x_n)| &= \left|\sum_{j=1}^\infty x_{n,j} y_j\right| \\
&\leq \sum_{j=1}^\infty |x_{n,j}||y_j| \\
&= \sum_{j=1}^J |x_{n,j}||y_j| + \sum_{j=J+1}^\infty |x_{n,j}||y_j| \\
&\leq \sum_{j=1}^J |x_{n,j}||y_j| + \sum_{j=J+1}^\infty |y_j| \\
&< \sum_{j=1}^J |x_{n,j}||y_j| + \frac{\varepsilon}{2}
\end{aligned}$$

(where we have used $x_n \in \bar{B}_E$ or $\sup_{j\geq 1} |x_{n,j}| \leq 1$). Now, since we have assumed

$$\lim_{n\to\infty} x_{n,j} = 0$$

for each j, we can conclude that

$$\lim_{n\to\infty} \sum_{j=1}^J |x_{n,j}||y_j| = 0$$

and so for n large enough

$$\sum_{j=1}^J |x_{n,j}||y_j| < \frac{\varepsilon}{2} \Rightarrow |\alpha(x_n)| < \varepsilon$$

This shows $\lim_{n\to\infty} \alpha(x_n) = 0$, as required.

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