MA3422 (Functional Analysis 2) Tutorial sheet 7
[March 16, 2017]

Name: Solutions

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1. Let p be a seminorm on a vector space E over $\mathbb{K}$ and let $\mathcal{T}_p$ be the topology on E generated by p.

   Show that $p\colon E \to \mathbb{R}$ is continuous (when E has the topology $\mathcal{T}_p$).

   [Hint: If $V \subseteq \mathbb{R}$ is open and $x_0 \in p^{-1}(V)$ we can take $p_0 = p(x_0) \in V$ and $r > 0$ such that $(p_0 - r, p_0 + r) \subset V$. If $x \in E$ has $p(x - x_0) < r$ then show $p(x) \in (p_0 - r, p_0 + r) \subset V \Rightarrow x \in p^{-1}(V)$. (A handy form of the triangle inequality for norms is $\|y - z\| \geq |\|y\| - \|z\||$ and it also holds for seminorms.)]

   Solution: Let $V \subseteq \mathbb{R}$ be open and we claim that $p^{-1}(V) \in \mathcal{T}_p$. For that, let $x_0 \in p^{-1}(V)$. Then $p(x_0) \in V$ and (since V is open in $\mathbb{R}$) there is $r > 0$ such that $(p(x_0) - r, p(x_0) + r) \subset V$. If $x \in B_p(x_0, r) = \{x \in E : p(x - x_0) < r\}$ then we have

   $$r > p(x - x_0) \geq |p(x) - p(x_0)|$$

   (by the triangle inequality) and so $p(x) \in (p(x_0) - r, p(x_0) + r) \subset V$, from which we have $x \in p^{-1}(V)$. This shows $B_p(x_0, r) \subset p^{-1}(V)$.

   Hence $B_p(x_0, r) \subset p^{-1}(V)$. Thus $p^{-1}(V) \in \mathcal{T}_p$.

   > **Aside.** About the triangle inequality. It is usually stated as $p(y + z) \leq p(y) + p(z)$ for $y, z \in E$, but the form $p(x - x_0) \geq |p(x) - p(x_0)|$ (for $x, x_0 \in E$) is equivalent.
   >
   > The inequality $p(x - x_0) \geq |p(x) - p(x_0)|$ is the same as one of two inequalities
   >
   > $$-(p(x) - p(x_0)) = p(x_0) - p(x) \leq -p(x - x_0) \text{ or } p(x - x_0) \geq p(x) - p(x_0)$$
   >
   > and that is the same as
   >
   > $$p(x_0) \leq p(x) + p(x - x_0) \text{ or } p(x) \leq p(x_0) + p(x - x_0)$$
   >
   > If you take $y = x$ and $z = x_0 - x$, the usual form of the triangle inequality gives
   >
   > $$p(x_0) = p(y + z) \leq p(y) + p(z) = p(x) + p(x_0 - x) = p(x) + p(x - x_0)$$
   >
   > and the other inequality follows by taking $y = x_0$, $z = x - x_0$ (and $y + z = x$ then).
   >
   > To show they are equivalent, note that $p(x - x_0) \geq |p(x) - p(x_0)| \Rightarrow p(x - x_0) \geq p(x) - p(x_0) \iff p(x) \leq p(x_0) + p(x - x_0)$. So, given $y, z \in E$ we can apply this with $x_0 = y$, $x = y + z$ to get $p(y + z) = p(x) \leq p(x_0) + p(x - x_0) = p(y) + p(z)$.

2. Write out a proof that $(c_0)^* = \ell^1$ (by which we mean that the elements $\alpha \in (c_0)^*$ are exactly those of the form $\alpha(x) = \alpha((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty x_n y_n$ for $y = (y_n)_{n=1}^\infty \in \ell^1$, and that $\|\alpha\| = \|y\|_1$).

   [Hint: There is a rather detailed proof that $(\ell^2)^* = \ell^2$ in Proposition 1.5.2]

   Solution: First if $y = (y_n)_{n=1}^\infty \in \ell^1$, it makes sense to define

$$\alpha_y \colon c_0 \to \mathbb{K} \text{ by the rule } \alpha_y(x) = \alpha_y((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty x_n y_n$$

   That is because

$$\sum_{n=1}^\infty |x_n y_n| = \sum_{n=1}^\infty |x_n||y_n| \leq \sum_{n=1}^\infty \|x\|_\infty |y_n| = \|x\|_\infty \sum_{n=1}^\infty |y_n| = \|x\|_\infty \|y\|_1 < \infty$$

   [we can instead say that this is true by Hölder's inequality with $p = \infty$ and $q = 1$] and so the series $\sum_{n=1}^\infty x_n y_n$ is guaranteed to be absolutely convergent (hence convergent in $\mathbb{K}$) for any choice of $x \in c_0$, $y \in \ell^1$.

   It is rather wasy to see that $\alpha_y \colon c_0 \to \mathbb{K}$ is linear (for each $y \in \ell^1$). In detail if $x = (x_n)_{n=1}^\infty \in c_0$, $x' = (x'_n)_{n=1}^\infty \in c_0$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned} \alpha_y(x + \lambda x') &= \alpha_y((x_n + \lambda x'_n)_{n=1}^\infty) = \sum_{n=1}^\infty (x_n + \lambda x'_n) y_n = \sum_{n=1}^\infty x_n y_n + \lambda \sum_{n=1}^\infty x'_n y_n \\ &= \alpha_y(x) + \lambda \alpha_y(x') \end{aligned}$$

   The inequality we have above tells us that

$$|\alpha_y(x)| = \left| \sum_{n=1}^\infty x_n y_n \right| \leq \sum_{n=1}^\infty |x_n y_n| \leq \|x\|_\infty \|y\|_1 = \|y\|_1 \|x\|_\infty$$

   and (since α_y is linear) it tells us that $\alpha_y \in (c_0)^*$ with $\|\alpha_y\| \leq \|y\|_1$.

   In fact we can show the reverse inequality $\|\alpha_y\| \geq \|y\|_1$. Fix y and for $N \geq 1$ define $x^{(N)} = (x_n^{(N)})_{n=1}^\infty = (x_1^{(N)}, x_2^{(N)}, \ldots, x_N^{(N)}, 0, 0, \ldots) \in c_0$ by

$$x_n^{(N)} = \begin{cases} \bar{y}_n / |y_n| & \text{if } y_n \neq 0 \text{ and } n \leq N \\ 0 & \text{if } y_n = 0 \text{ or } n > N \end{cases}$$

   Then $\|x^{(N)}\|_\infty = \sup_n |x_n^{(N)}| \leq 1$ and

$$\alpha_y(x^{(N)}) = \sum_{n=1}^\infty x_n^{(N)} y_n = \sum_{n=1}^N x_n^{(N)} y_n = \sum_{n=1}^N |y_n|$$

But $|\alpha_y(x^{(N)})| \leq \|\alpha_y\| \|x^{(N)}\|_\infty \leq \|\alpha_y\|$ and so $\sum_{j=1}^N |y_n| \leq \|\alpha_y\|$ for each N. Let $N \to \infty$ to get

$$\|y\|_1 = \sum_{n=1}^{\infty} |y_n| \leq \|\alpha_y\|\|.$$

We now set about showing that $\alpha \in (c_0)^*$ implies there is $y \in \ell^1$ with $\alpha = \alpha_y$.

Consider the ("standard basis") vectors $e_1 = (1, 0, 0, \ldots) \in c_0$, $e_2 = (0, 1, 0, 0, \ldots) \in c_0$ and in general $e_n = (\delta_{j,n})_{j=1}^\infty$ (using Dirac δ notation). Define $y_n = \alpha(e_n)$. [These values must work if there is any y with $\alpha = \alpha_y$.]

We claim that $y = (y_n)_{n=1}^\infty \in \ell^1$. For that, define $x^{(N)}$ exactly as above (in terms of y and N) so that

$$x^{(N)} = \sum_{n=1}^{N} x_n^{(N)} e_n \in c_0$$

and $\|x^{(N)}\| \leq 1$. Since α is linear,

$$\alpha(x^{(N)}) = \alpha\left(\sum_{n=1}^{N} x_n^{(N)} e_n\right) = \sum_{n=1}^{N} x_n^{(N)} \alpha(e_n) = \sum_{n=1}^{N} |y_n|.$$

But also $|\alpha(x^{(N)})| \leq \|\alpha\| \|x^{(N)}\| \leq \|\alpha\|$. Thus

$$\sum_{n=1}^{N} |y_n| \leq \|\alpha\|$$

for each N. Hence $\sum_{n=1}^\infty |y_n| \leq \|\alpha\|$, and $y \in \ell^1$.

By linearity $\alpha(x) = \alpha_y(x)$ for any finitely nonzero $x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) = \sum_{n=1}^N x_j e_j \in c_0$. These finitely nonzero x's are dense in c_0, α and α_y are continuous, they agree on the dense set and so $\alpha = \alpha_y$.

Maybe an explanation of why the finitely nonzero x's are dense in c_0. If $x = (x_n)_{n=1}^\infty \in c_0$, we know $\lim_{n\to\infty} x_n = 0$. So if $\varepsilon > 0$ is arbitrary, then there is N so that $n > N \Rightarrow |x_n| < \varepsilon$. Then

$$x - \sum_{n=1}^{N} x_n e_n = x - (x_1, \ldots, x_N, 0, 0, \ldots) = (0, 0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots)$$

so that

$$\left\| x - \sum_{n=1}^{N} x_n e_n \right\| = \sup_{n>N} |x_n| = \max_{n>N} |x_n| < \varepsilon.$$

This shows that $x \in c_0$ and $\varepsilon > 0$ implies there is a finitely nonzero $\tilde{x}$ with $\|x - \tilde{x}\|_\infty < \varepsilon$. That is the finitely nonzero sequences are dense in c_0.

It also shows that

$$x \in c_0 \Rightarrow x = \sum_{n=1}^{\infty} x_n e_n = \lim_{N \to \infty} \sum_{n=1}^{N} x_n e_n.$$

Moreover the only way to have $x = \sum_{n=1}^{\infty} a_n e_n$ for scalars a_n is to take $a_n = x_n$. This kind of property for the vectors e_n has a name. It means that $(e_n)_{n=1}^{\infty}$ is called a *Schauder basis* for the Banach space c_0.

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