MA3422 (Functional Analysis 2) Tutorial sheet 6
[March 9, 2017]

Name:

1. Let E be a normed space and $F \subseteq E$ a linear subspace (normed by restricting the norm of E to F). Show that the restriction map

$$: E^* \to F^* : \alpha \mapsto \alpha|_F$$

is surjective.

Solution:

If α is a continuous linear functional on E, then its restriction to the subspace F will certainly be continuous and linear. So the restriction map makes sense.

The fact that the map is surjective follows directly from (a corollary of) the Hahn-Banach theorem. If $\alpha \in F^*$, then α is linear from F to $\mathbb{K}$ and satisfies $|\alpha(x)| \leq \|\alpha\|_{F^*}\|x\|$ for all $x \in F$. So, by Hahn-Banach (applied to the seminorm $p(x) = \|\alpha\|_{F^*}\|x\|$ for $x \in E$), there is a linear extension $\beta\colon E \to \mathbb{K}$ with $|\beta(x)\| \leq \|\alpha\|_{F^*}\|x\|$ for all $x \in E$.

This β is then in E^* and satisfies $\beta|_F = \alpha$. So the restriction map is surjective.

2. Let E be a normed space and $F \subseteq E$ a dense linear subspace (normed by restricting the norm of E to F). Show that the restriction map

$$: E^* \to F^* : \alpha \mapsto \alpha|_F$$

is surjective, norm preserving and bijective.

Solution: (#1)

By the previous question, the map is surjective. Every $\alpha \in F^*$ extends to some $\beta \in E^*$.

In fact $\|\beta\|_{E^*} = \|\alpha\|_{F^*}$. (We have $\|\beta\|_{E^*} \leq \|\alpha\|_{F^*}$ from the use of Hahn-Banach above and since β extends α we must have $\|\beta\|_{E^*} \geq \|\alpha\|_{F^*}$ — because if $F \neq \{0\}$, then $\|\beta\|_{E^*} = \sup_{x \in E\setminus\{0\}} |\beta(x)|/\|x\| \geq \sup_{x \in F\setminus\{0\}} |\beta(x)|/\|x\| = \sup_{x \in F\setminus\{0\}} |\alpha(x)|/\|x\| = \|\alpha\|_{F^*}$.)

In fact, since F is dense in E, we have that β is uniquely determined by α — if two continuous $\mathbb{K}$-valued functions agree on a dense set then they have to be equal.

This shows that $\|\beta\|_{E^*} = \|\beta|_F\|_{F^*}$ for all $\beta \in E^*$ (because for $\alpha = \beta|_F \in F^*$, the unique extension is the original β).

> A more fancy argument would be that the restriction map is a linear map (rather easy to verify). Since F is dense in E, if $x \in E$ has $\|x\| < 1$, there is a sequence $(x_n)_{n=1}^\infty$ in F with $\lim_{n\to\infty} x_n = x$ and $\|x_n\| < 1$ for each n. We have $|\beta(x_n)| = |\alpha(x_n)| \leq \|\alpha\|_{F^*}$ for each n and (by continuity of β)
>
> $$|\beta(x)| = \lim_{n\to\infty} |\beta(x_n)| \leq \|\alpha\|_{F^*}$$

Thus $\|\beta\| \leq \|\alpha\|_{F^*}$.

It is easy that $\|\beta\| \geq \|\alpha\|_{F^*}$. So $\|\beta\| = \|\alpha\|_{F^*}$.

A linear surjective norm-preserving map has zero kernel. Hence bijective.

Solution: (#2 without Hahn-Banach)

As before the restriction map makes sense.

If $\alpha \in F^*$, then we know that α is uniformly continuous on F. It follows then that α has a unique continuous extension to the closure E. (The idea is that α maps Cauchy sequences $(x_n)_{n=1}^\infty$ in F to Cauchy sequences $(\alpha(x_n))_{n=1}^\infty$ in $\mathbb{K}$, and so $\lim_{n\to\infty} \alpha(x_n)$ exists in $\mathbb{K}$. If we take a sequence $(x_n)_{n=1}^\infty$ in F that converges to a limit $x \in E$ we can then try to define $\beta(x) = \lim_{n\to\infty} \alpha(x_n)$. That limit will exist because $(x_n)_{n=1}^\infty$ will be Cauchy in E and so in F. We need to check that if $(y_n)_{n=1}^\infty$ is a different sequence in F that converges to the same limit $x \in E$, then $\lim_{n\to\infty} \alpha(y_n) = \lim_{n\to\infty} \alpha(x_n)$ (so that our definition of $\beta(x)$ does not depend on the sequence chosen), but that follows because $x_1, y_1, x_2, y_2, x_3, \ldots$ is also a sequence in F converging to the same x and so $\alpha(x_1), \alpha(y_1), \alpha(x_2), \alpha(y_2), \ldots$ has a limit in $\mathbb{K}$. So we can define the extension $\beta \colon E \to \mathbb{K}$. It is an extension because if $x \in F$ we can take $x_n = x$ for all n and so $\beta(x) = \alpha(x)$. To show β is continuous we can check that it is Lipschitz with the same Lipschitz constant as α. If $x, z \in E$, choose sequences $(x_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ in F with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} z_n = z$. Since

$$|\alpha(x_n) - \alpha(z_n)| \leq \|\alpha\|_{F^*} \|x_n - z_n\|$$

holds for all n we can let $n \to \infty$ to get $|\beta(x) - \beta(z)| \leq \|\alpha\|_{F^*} \|x - z\|$. This is all a standard result, which we might just quote. We need $\mathbb{K}$ to be complete.)

We can also check that β is linear. If we take $x, z \in E$ and $\lambda \in \mathbb{K}$ and then sequences $(x_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ in F as we just did then $\lim_{n\to\infty} x_n + \lambda z_n = x + \lambda z$ and so

$$\beta(x + \lambda z) = \lim_{n\to\infty} \alpha(x_n + \lambda z_n) = \lim_{n\to\infty} \alpha(x_n) + \lambda\alpha(z_n) = \beta(x) + \lambda\beta(z)$$

We can also see now that we have got as much as we got from the Hahn-Banach theorem because we have that the Lipschitz constant of β is no bigger than that of α, so the norm of β is also no bigger than $\|\alpha\|_{F^*}$.

We could finish off the norm preseerving part as before or we could take $x \in E \setminus \{0\}$ and choose a sequence $(x_n)_{n=1}^\infty$ in $F \setminus \{0\}$ with $\lim_{n\to\infty} x_n = x$. Then

$$\frac{|\beta(x)|}{\|x\|} = \lim_{n\to\infty} \frac{|\alpha(x_n)|}{\|x_n\|} \leq \|\alpha\|_{F^*}.$$

It follows that $\|\beta\|_{E^*} \leq \|\alpha\|_{F^*}$. We can notice again that $\|\beta\|_{E^*} \geq \|\alpha\|_{F^*}$ is easy.

Richard M. Timoney