MA3422 (Functional Analysis 2) Tutorial sheet 5

[February 23, 2017]

Name: Solutions

1. Let E be a Banach space and recall $\mathcal{B}(E)$ is a Banach algebra. If $T \in \mathcal{B}(E)$ has $\|T\| < 1$, show that $S = \sum_{n=0}^\infty T^n = \mathrm{id}_E + T + T^2 + \cdots$ converges to some $S \in \mathcal{B}(E)$ and that $S(\mathrm{id}_E - T) = \mathrm{id}_E = (\mathrm{id}_E - T)S$. [Hint: absolutely convergent series in a Banach space are convergent.]

   Solution: We know $\|T^2\| \leq \|T\|\|^2$, $\|T^3\| = \|T^2T\| \leq \|T^2\|\|T\| \leq \|T^3\|$ and (by induction on n) we can see that $\|T^n\| \leq \|T\|^n$ for $n = 1, 2, \ldots$. For $n = 0$, we could agree that $T^0 = \mathrm{id}_E$ and that $\|T\|^0$ is to mean 1, and then we even have $\|T^n\| \leq \|T\|^n$ for $n = 0$ also.

   So the series for S is $\sum_{n=0}^\infty T^n$ and it is abssolutely convergent because

   $$\sum_{n=0}^\infty \|T^n\| \leq \sum_{n=0}^\infty \|T\|^n = \frac{1}{1 - \|T\|} < \infty$$

   (geometric series with ration < 1 (in absolute value)).

   Since $\mathcal{B}(E)$ is a Banach space, we can conclude that

   $$S = \sum_{n=0}^\infty T^n = \lim_{n\to\infty} \mathrm{id}_E + T + T^2 + \cdots + T^n$$

   exists in $\mathcal{B}(E)$.

   It remains to justify the assertions that $S(\mathrm{id}_E - T) = \mathrm{id}_E = (\mathrm{id}_E - T)S$.

   To make this a convincing argument, observe that left (and right) multiplication by T gives a bounded (continuoous) linear operator on $\mathcal{B}(E)$. That is, if we define $L_T \colon \mathcal{B}(E) \to \mathcal{B}(E)$ by $L_T(U) = TU$, then L_T is linear (by the algebra properties of $\mathcal{B}(E)$) and the Banach algebra property of submultiplicativity of the norm gives $\|L_T(U)\| = \|TU\| \leq \|T\|\|U\|$ for each $U \in \mathcal{B}(E)$. So $\|L_T\| \leq \|T\|$ and L_T is bounded.

   The same argument applies to $R_T(U) = UT$ (right multiplication).

   Thus

   $$S(\mathrm{id}_E - T) = S - ST = \lim_{n\to\infty} \mathrm{id}_E + T + T^2 + \cdots + T^n - (\lim_{n\to\infty} \mathrm{id}_E + T + T^2 + \cdots + T^n)T$$

   From continuity of R_T we can bring the T inside the limit and get

   $$\begin{aligned} &= \lim_{n\to\infty} \mathrm{id}_E + T + T^2 + \cdots + T^n - \lim_{n\to\infty} ((\mathrm{id}_E + T + T^2 + \cdots + T^n)T) \\ &= \lim_{n\to\infty} \mathrm{id}_E + T + T^2 + \cdots + T^n - \lim_{n\to\infty} (T + T^2 + \cdots + T^{n+1}) \\ &= \lim_{n\to\infty} \mathrm{id}_E - T^{n+1} \end{aligned}$$

(by the lmit of a difference). This is just id_E since we know $\|T^{n+1}\| \leq \|T\|^{n+1} \to 0$ as $n \to \infty$.

The argument for $(\mathrm{id}_E - T)S = \mathrm{id}_E$ is just the same, but using L_T instead of R_T. It begins

$$(\mathrm{id}_E - T)S = S - TS.$$

Aside

This little fact is usually proved as one of the first lemmas for spectral theory of (bounded) linear operators T on (complex) Banach spaces E.

The conclusion can be stated that S is an inverse (multiplicative inverse) for $\mathrm{id}_E - T$ in $\mathcal{B}(E)$.

The *spectrum* of $T \in \mathcal{B}(E)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda \mathrm{id}_E - T) \text{ is } \textbf{not} \text{ invertible in } \mathcal{B}(E)\}$$

While you could define this for the case of a Banach space E over $\mathbb{R}$ in just the same way (replace $\lambda \in \mathbb{C}$ by $\lambda \in \mathbb{R}$), quite a few things don't work so well over $\mathbb{R}$. (In the case of finite dimensional E, we can choose a basis and reprsent T by a square matrix A. Then $\sigma(T)$ is the set of eigenvalues of A. You may recall that real matrices may not have real eigenvalues, but they will have complex ones.)

So what our problem shows (in this language) is

$$\|T\| < 1 \Rightarrow \sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

and we could rescale T in order to show more generally

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$$

But the topic of spectral theory is one of many that we will not have time to investigate.

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2. Let G be a group. Show that G contains a maximal abelian subgroup. [Hint: a subgroup $H \subseteq G$ is a subset that contains the identity element e_G of G and satisfies $x, y \in H \Rightarrow xy^{-1} \in H$. These properties summarise that H is also a group with the same multiplication as G. To say that H is abelian means that $x, y \in H \Rightarrow xy = yx$.]

Solution:

First a digression, or an expansion of the Hint.

We should perhaps recall the relevant definitions before we proceed to the solution itself.

A *group* G is a set with a binary operation $: G \times G \to G$ (which we call multiplication and where we denote the image of $(g_1, g_2) \in G \times G$ (or the product of g_1 times g_2 in usual terminology) by $g_1 g_2$. The group is supposed to satisfy additional properties:

(i) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ (for all $g_1, g_2, g_3 \in G$, the associative law);

(ii) there exists an element $e \in G$ so that $eg = ge = g$ for all $g \in G$ (existence of identity element);

(iii) for each $g \in G$ there exists $h \in G$ with $gh = hg = e$ (existence of inverses).

A group G is called *abelian* if it satisfies $g_1 g_2 = g_2 g_1$ for each $g_, g_2 \in G$.

If G is a group, a subset $H \subseteq G$ is called a *subgroup* of G if G becomes a group in the same product operation as G (restricted to H). The subgroup H cannot be the empty set (because groups are never empty — they have identity elements). The identity element of H must be the identity of G (because if $f \in H$ is the identity of H, then $ff = f$. Multiply both sides by the inverse of f in G to get $f = e =$ the identity element of G. It follows then that the inverse of $h \in H$ must be the same whether we consider it as the inverse in the subgroup or the inverse in G.

So subgroups have to be nonempty subsets of G (in fact they must contain the identity e of G) and we must have two other properties for $H \subset G$ to be a subgroup:

$$h_1, h_2 \in H \Rightarrow h_1 h_2 \in H \text{ and } h \in H \Rightarrow h^{-1} \in H.$$

These can be combined into an equivalent property:

$$h_1, h_2 \in H \Rightarrow h_1 h_2^{-1} \in H.$$

So a subgroup is $H \subseteq G$ with $e \in H$ and this property.

An abelian subgroup is a subset $H \subset G$ with the properties:

$$e \in H, \quad h_1, h_2 \in H \Rightarrow h_1 h_2^{-1} \in H \text{ and } h_1 h_2 = h_2 h_1.$$

Notice that if the whole group G is abelian, then $H = G$ is an abelian subgroup of itself and it is the maximal (in fact the largest) abelian subgroup as all subgroups are certainly contained in G. However, it does not really help the proof with Zorn's lemma to notice that this is so.

To apply Zorn's lemma we consider the set S of all abelian subgroups of G, ordered by set inclusion $\subseteq$. So

$$S = \{H : H \text{ a subgroup of } G, H \text{ abelian}\}$$

and the partial order on S is $\subseteq$.

We know that $S \neq \emptyset$ because $H = \{e\}$ is always an abelian subgroup of G (e the identity element of G).

To apply Zorn's lemma, we need to show that each nonempty chain $C \subset S$ has an upper bound in S. Put

$$K = \bigcup_{H \in C} H.$$

We claim that $K \in S$.

It is clear that $K \subseteq G$ since $H \subseteq G$ for each $H \in C$. Also $e \in K$ since $C \neq \emptyset \Rightarrow \exists H \in C \Rightarrow e \in H \subseteq K$. We have to show that for any $k_1, k_2 \in K$ we must have $k_1 k_2^{-1} \in K$ and $k_1 k_2 = k_2 k_1$.

So fix $k_1, k_2 \in K$. Then there exist $H_1, H_2 \in C$ with $k_1 \in H_1$ and $k_2 \in H_2$. As C is a chain, we have $H_1 \subseteq H_2$ or else $H_2 \supseteq H_1$. In the case $H_1 \subseteq H_2$, let $H = H_2$ and in the other case $H_2 \supseteq H_1$, let $H = H_1$. Now $H \in C$ and $k_1, k_2 \in H$. As H is an abelian subgroup of G, we know $k_1 k_2^{-1} \in H \subseteq K$ and $k_1 k_2 = k_2 k_1$. Thus $K \subset G$ is an abelian subgroup of G and so $K \in S$.

As $H \subseteq K$ holds for each $H \in C$, we have that K is an upper bound for C.

Thus we have shown that every chain C in S has an upper bound. By Zorn, S has a maximal element — which means that there is a maximal abelian subgroup of G.

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