MA3422 (Functional Analysis 2) Tutorial sheet 4
[February 16, 2017]

Name: Solutions

1. Let $E = \ell^2 \cap \ell^1$ with the norm $\|\cdot\|_2$ and $F = \ell^1$. Define $T\colon E \to F$ by $Tx = x$. Show that T is not bounded but has closed graph. [Hint: consider $x = (1, 1, \dots, 1, 0, 0, \dots)$ with n ones and the rest of the terms 0 to show that T is not bounded.]

   Solution: Solution: As in the hint consider $x_n = (1, 1, \dots, 1, 0, 0, \dots)$ with n ones. Then $\|x_n\|_2 = \sqrt{1^2 + 1^2 + \cdots 1^2 + 0} = \sqrt{n}$ while $\|Tx_n\| = \|x_n\|_1 = 1 + 1 + \cdots + 1 + 0 = n$. There is no finite constant C with $n \leq C\sqrt{n}$ for all n and hence T is not bounded.

   Suppose now (x_n, y_n) is in the graph of T for all n (different x_n from above) and $(x_n, y_n) \to (x, y)$ in $E \times F$ (as $n \to \infty$). So $y_n = Tx_n = x_n$. Also $x_n \to x$ in E and $y_n = x_n \to y$ in F. So $\|x_n - x\|_2 \to 0$ as $n \to \infty$ and $\|x_n - y\|_1 \to 0$ as $n \to \infty$.

   Then $x \in \ell^2$ and $y \in \ell^1$. But also, the k^{th} term of x_n converges (in $\mathbb{K}$) to the k^{th} term of x (for each $k \in \mathbb{N}$) and to the k^{th} term of y.

   That is because if $x_{n,k}$ is the k^{th} term of x_n and $x_{\infty,k}$ is the k^{th} term of x, then $|x_{n,k} - x_{\infty,k}| \leq \|x_n - x\|_2$ and a similar fact holds for the norm $\|\cdot\|_1$.

   So $x = y \in \ell^2 \cap \ell^1$. That means $(x, y) = (x, x)$ is in the graph of T and thus the graph is closed.

   > An alternative argument can be based on the fact that $\ell^1 \subset \ell^2$ and the inclusion map is continuous (bounded linear, even norm one — see Examples 2.6.6 in MA3421 notes). So since $x_n \to x$ in ℓ^1, it must be that $x_n \to x$ in ℓ^2, or $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. As $x_n \to y$ in ℓ^2 norm, we must have $x = y$.

   Why does this not contradict the closed graph theorem?

   Solution: The closed graph theorem requires E and F to be Banach spaces, and T to be a linear operator. It says that if the graph of T is closed, then T must be bounded — which is not the case here.

   We know that T is linear and F is Banach and so it must be that E is not Banach, not complete.

   > [I think we don't need to explain that in detail.
   > We can view E as a linear subspace of ℓ^2 (and we use the norm $\|\cdot\|_2$). For E to be complete in the norm of ℓ^2, it would have to be closed in ℓ^2. But it is not closed — in fact it is dense in ℓ^2 since it contains all finitely nonzero sequences. So if it were closed we would have $E = \ell^2 \cap \ell^1 = \ell^2$, so that $\ell^2 \subseteq \ell^1$ — which is false. An example is $(1/n^{2/3})_{n=1}^\infty \in \ell^2$ because $\sum_{n=1}^\infty 1/n^{4/3} < \infty$ but is not in ℓ^1 because $\sum_{n=1}^\infty 1/n^{2/3} = \infty$.]

2. If E and F are Banach spaces and $T\colon E \to F$ is linear, we define the *separating space* S_T of T by

$$S_T = \{y \in F : \exists (x_n)_{n=1}^\infty \text{ in } E \text{ with } \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} T(x_n) = y\}$$

Show that if T has closed graph then $S_T = \{0\}$.

Solution: If T is bounded, it is continuous and in particular continuous at $0 \in E$. So if $(x_n)_{n=1}^\infty$ is a sequence in E with $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} T(x_n) = T(0) = 0$. Hence the only possible element of S_T is 0.

We do indeed have $0 \in S_T$ because we can consider the constant sequence $x_n = 0$ (all $n \geq 1$). It satisfies $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(0) = T(0) = 0$, so that $0 \in S_T$. [In fact any linear transformation T would have $0 \in S_T$, whether T is continuous or not.]

Aside: In fact for any linear operator $T\colon E \to F$ between normed spaces, S_T is a closed linear subspace of F (something that is not so hard to verify) and the same proof as above shows that $S_T = \{0\}$ if T is bounded.

It is also true (and fairly elementary to prove using linearity) that T has closed graph if and only if $S_T = \{0\}$.

---

3. If E is a normed space, a *projection* on E means a bounded linear operator $P\colon E \to E$ such that $P \circ P = P$.

Show that if P is a projection on E, then $E = P(E) \oplus \ker P$, a direct sum of closed subspaces, and that there is a constant $C > 0$ (depending on P) such that for $y \in P(E)$, $z \in \ker P$,

$$\|y + z\| \leq \|y\| + \|z\| \leq C(\|y + z\|)$$

holds.

Solution: As the range and kernel of a linear transformation we know that $P(E)$ and $\ker P$ are vector subspaces of E.

For $x \in E$ we can write $x = P(x) + (x - P(x))$. Then $P(x) \in P(E)$ is clear and we can check

$$P(x - P(x)) = P(x) - (P \circ P)(x) = P(x) - P(x) = 0 \Rightarrow x - P(x) \in \ker P.$$

Thus $E = P(E) + \ker P$. It is a direct sum since if $y = z$ for $y \in P(E)$ and $z \in \ker P$, then $y = P(t)$ for some $t \in E$ and so

$$P(P(t)) = P(t) \Rightarrow P(y) = y \Rightarrow P(z) = y \Rightarrow 0 = y.$$

Thus $P(E) \cap \ker P = \{0\}$ and $E = P(E) \oplus \ker P$ (a vector space direct sum)[1].

[1] The idea is then that each $x \in E$ must have a unique expression as $x = y + z$ with $y \in P(E)$ and $z \in \ker P$ because if we have another such expression $x = y' + z'$ for x, then $y - y' = z' - z \in P(E) \cap \ker P = \{0\} \Rightarrow y = y'$ and $z = z'$.

The subspace $\ker P$ is clearly closed (being the inverse image of $\{0\}$ under the continuous map P) and we can argue that $P(E)$ is closed because if $(y_n)_{n=1}^{\infty}$ is a sequence in $P(E)$ that converges to a limit $x \in E$, then

$$x = \lim_{n\to\infty} y_n = \lim_{n\to\infty} P(y_n) = P(x)$$

(by continuity of P) and so $x \in P(E)$.

As regards the inequalities $\|y+z\| \leq \|y\| + \|z\|$ holds by the triangle inequality. Since P is a bounded linear operator there is $C > 0$ so that $\|P(x)\| \leq C\|x\|$ for each $x \in E$. Thus

$$\|x - P(x)\| \leq \|x\| + \|P(x)\| \leq (C+1)\|x\|.$$

Taking $x = y + z$ with $y \in P(E)$, $z \in \ker P$ we have

$$\|y\| + \|z\| = \|P(x)\| + \|x - P(x)\| \leq (2C+1)\|x\| = (2C+1)\|y+z\|.$$

Aside: If a Banach space E is a vector space direct sum of two closed subspaces F and G, then it means that each $x \in E$ can be expressed in a unique way as $x = y + z$ with $y \in F$ and $z \in G$.

So the map $T \colon F \oplus_1 G \to E$ defined by $T(y,z) = y + z$ is a bijection. We can see easily that T is linear and bounded as $\|T(y,z)\|_E = \|y+z|_E \leq \|y\|_E + \|z\|_E = \|(y,z)\|_1$. Since F and G are closed in E, and E is complete, F and G are Banach spaces, and so is $F \oplus_1 G$. By the inverse mapping theorem corollary to the open mapping theorem, T^{-1} is also bounded.

That is $\|T^{-1}(x)\|_1 \leq \|T^{-1}\|\|x\|$ for $x \in E$. If we write $x = y + z$ with $y \in F$ and $z \in G$, then $T^{-1}(x) = (y,z)$ and we have

$$\|(y,z)\|_1 = \|y\| + \|z\| \leq \|T^{-1}\|\|y+z\|$$

for $y \in F$ and $z \in G$.

We can dsefine a projection $P \colon E \to E$ by

$$P(y+z) = y \qquad (y \in F, z \in G)$$

and then it follows that P is linear, $P(E) = F$, $\ker P = G$ and $\|P\| \leq \|T^{-1}\|$.

Richard M. Timoney