MA3422 (Functional Analysis 2) Tutorial sheet 3
[February 9, 2017]

Name: Solutions

On this sheet H denotes a separable Hilbert space.

For $T \in \mathcal{B}(H)$, we define T to be *positive* (or sometimes called positive semidefinite) if it is true that $\langle Tx, x\rangle \geq 0$ for each $x \in H$. [So that means $\langle Tx, x\rangle$ is real and nonnegative.] We write $T \geq 0$ in this case.

1. Show that $T \in \mathcal{B}(H) \Rightarrow T^*T \geq 0$

   Solution: For $x \in H$, we have

   $$\langle (T^*T)x, x\rangle = \langle T^*(Tx), x\rangle = \overline{\langle x, T^*(Tx)\rangle} = \overline{\langle Tx, Tx\rangle} = \|Tx\|^2 \geq 0.$$

   (You could also use $(T^*)^* = T$, which is established by $\langle T^*x, y\rangle = \overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$. Then you can say $\langle T^*(Tx), x\rangle = \langle Tx, Tx\rangle \geq 0$.)

2. Show that $T \in \mathcal{B}(H)$ and $T \geq 0$ implies $T^* = T$. [Hint: consider $\langle T(x \pm y), x \pm y\rangle$ and $\langle T(x \pm iy), x \pm iy\rangle$, for $x, y \in H$. Review the proof of the parallelogram identity. Show $\operatorname{Im}\langle Tx, y\rangle = \operatorname{Im}\langle x, Ty\rangle$ and the same for the real parts.]

   Solution: We have

   $$\langle T(x+y), x+y\rangle = \langle Tx + Ty, x+y\rangle = \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle$$

   $$\langle T(x-y), x-y\rangle = \langle Tx - Ty, x-y\rangle = \langle Tx, x\rangle - \langle Tx, y\rangle - \langle Ty, x\rangle + \langle Ty, y\rangle$$

   So both are positive and we can subtract to get

   $$2(\langle Tx, y\rangle + \langle Ty, x\rangle) \in \mathbb{R}$$

   and so

   $$\operatorname{Im}(\langle Tx, y\rangle + \langle Ty, x\rangle) = 0.$$

   This gives

   $$\operatorname{Im}(\langle Tx, y\rangle - \langle x, Ty\rangle) = 0$$

   since $\langle Ty, x\rangle = \overline{\langle x, Ty\rangle}$.

   > [Actually we could manage it another way, without subtracting, because two of the terms on the right of the expansion of $\langle T(x+y), x+y\rangle$ are also positive, and therefore real. They are $\langle Tx, x\rangle$ and $\langle Ty, y\rangle$.]

So we have $\operatorname{Im}\langle Tx, y\rangle = \operatorname{Im}\langle x, Ty\rangle$.

We can repeat with iy instead of y, or just replace y by iy to get

$$\operatorname{Im}\langle Tx, iy\rangle = \operatorname{Im}\langle x, iTy\rangle.$$

That is

$$\operatorname{Im}(-i\langle Tx, y\rangle) = \operatorname{Im}(-i\langle x, Ty\rangle),$$

and gives

$$\operatorname{Re}(-\langle Tx, y\rangle) = \operatorname{Re}(-\langle x, Ty\rangle).$$

So $\operatorname{Re}\langle Tx, y\rangle = \operatorname{Re}\langle x, Ty\rangle$.

Combining the real and imaginary parts, we must have

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

As this holds for all $x, y \in H$, it means that $T^* = T$.

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