MA3422 (Functional Analysis 2) Tutorial sheet 2
[February 2, 2017]

Name: Solutions

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1. In $L^2[0,1]$ take $f(t) = t$ and compute the orthogonal projection of f onto

$$\mathrm{span}\{1, e^{2\pi i t}, e^{-2\pi i t}\}$$

Solution: Recall that the functions $\phi_0(t) = 1$, $\phi_1(t) = e^{2\pi i t}$ and $\phi_{-1}(t) = e^{-2\pi i t}$ are orthonormal vectors in $L^2[0,1]$.

> [If we want to check that we could note that $|\phi_n(t)| = 1$ for $t \in [0,1]$ so that $\|\phi_n\|_2^2 = \int_{[0,1]} |\phi_n(t)|^2\,dt = 1$. Also, if $n \neq m$, then $\phi_n(t)\overline{\phi_m(t)} = e^{2\pi i(n-m)t}$ and so $\langle \phi_n, \phi_m \rangle = \int_{[0,1]} \phi_n(t)\overline{\phi_m(t)}\phi_n(t)\overline{\phi_m(t)}\,dt = \left[\frac{1}{2\pi i(n-m)t} e^{2\pi i(n-m)t}\right]_0^1 = 0$.]

Since then we already have an orthonormal baseis for the span, we do not need to apply Gram-Schmidt to get one.

We know that the orthogonal projection of f onto the span is

$$g = \langle f, \phi_0\rangle\phi_0 + \langle f, \phi_1\rangle\phi_1 + \langle f, \phi_{-1}\rangle\phi_{-1}$$

We compute

$$\begin{aligned}
\langle f, \phi_0 \rangle &= \int_{[0,1]} f(t)\overline{\phi_0(t)}\,dt \\
&= \int_{[0,1]} t\,dt \\
&= [t^2/2]_0^1 = 1/2 \\
\langle f, \phi_1 \rangle &= \int_{[0,1]} f(t)\overline{\phi_1(t)}\,dt \\
&= \int_{[0,1]} te^{-2\pi i t}\,dt \\
&= \int_{[0,1]} td\left(\frac{e^{-2\pi i t}}{-2\pi i}\right)\,dt \\
&= \left[t\left(\frac{e^{-2\pi i t}}{-2\pi i}\right)\right]_0^1 - \int_{[0,1]} \frac{e^{-2\pi i t}}{-2\pi i}\,dt = 1/(-2\pi i) - 0 = i/(2\pi)
\end{aligned}$$

$$\begin{aligned}
\langle f, \phi_{-1} \rangle &= \int_{[0,1]} f(t)\overline{\phi_{-1}(t)}\, dt \\
&= \int_{[0,1]} te^{2\pi i t}\, dt \\
&= \left[t\left(\frac{e^{2\pi i t}}{2\pi i} \right) \right]_0^1 - \int_{[0,1]} \frac{e^{2\pi i t}}{2\pi i}\, dt = 1/(2\pi i) - 0 = -i/(2\pi)
\end{aligned}$$

Thus

$$\begin{aligned}
g(t) &= \frac{1}{2}1 + \frac{i}{2\pi}e^{2\pi i t} - \frac{i}{2\pi}e^{-2\pi i t} \\
&= \frac{1}{2} + \frac{i}{2\pi}(\cos(2\pi t) + i\sin(2\pi t)) - \frac{i}{2\pi}(\cos(2\pi t) - i\sin(2\pi t)) \\
&= \frac{1}{2} - \frac{1}{\pi}\sin(2\pi t))
\end{aligned}$$

How would you find is the (shortest) distance from f to that subspace? [You need not actually compute it.]

Solution: As g is the closest point in the span to f, we should compute $\|f - g\|_2$.

Since we have simplified g above, this is not so difficult to compute in fact. It is

$$\begin{aligned}
\|f - g\|_2 &= \sqrt{\int_{[0,1]} (t - 1/2 + (1/\pi)\sin(2\pi t))^2\, dt} \\
&= \sqrt{\int_{[0,1]} 1/4 - t + t^2 - (1/\pi)\sin(2\pi t) + (2t/\pi)\sin(2\pi t) + 1/\pi^2 \sin^2(2\pi t)\, dt}
\end{aligned}$$

and that is $\sqrt{1/12 - 1/(2\pi^2)}$.

2. Let $1 \leq p \leq \infty$ with $p \neq 2$. Show that ℓ^p is not a Hilbert space. [Hint: try the parallelogram identity for e_1 and e_2.]

   Solution: For $1 \leq p < \infty$ we have

$$\begin{aligned}\|e_1 \pm e_2\|_p &= \|(1, \pm 1, 0, 0, \ldots)\|_p \\ &= (1^p + |-1|^p + 0)^{1/p} = 2^{1/p}\end{aligned}$$

   while for $p = \infty$,

$$\begin{aligned}\|e_1 \pm e_2\|_\infty &= \|(1, \pm 1, 0, 0, \ldots)\|_\infty \\ &= \sup(1, |-1|, 0) = 1.\end{aligned}$$

   For all p we find $\|e_1\|_p = 1 = \|e_2\|_p$ and so

$$\|e_1 + e_2\|_p^2 + \|e_1 - e_2\|_p^2 = 2(2^{2/p}) \neq 4 = 2(\|e_1\|_p^2 + \|e_2\|_p^2)$$

   if $p \neq 2$. (For short, we have included $p = \infty$ with $2^{2/\infty}$ meaning 1.)

   So the parallelogram identity $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ fails to hold for $x = e_1$ and $y = e_2$ (and the norm $\|\cdot\|_p$ for ℓ^p) if $p \neq 2$.

   Thus there is no inner product on ℓ^p giving rise to the norm $\|\cdot\|_p$.

   [**Aside:** A stronger result is that there is no equivalent norm on ℓ^p for which it is a Hilbert space, but that is much harder to establish.]

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