MA3422 (Functional Analysis 2) Tutorial sheet 1
[January 26, 2017]

Name: Solutions

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1. If V is an inner product space, show that the triangle inequality holds (for the norm arising from the inner product). [Hint: Use Cauchy-Schwarz.]

   Solution: For $x, y \in V$ we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &\quad \text{(using Cauchy-Schwarz)} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

   Taking square roots we get $\|x+y\| \le \|x\| + \|y\|$.

2. In $H = L^2[0,1]$ use the Gram-Schmidt process to find an orthonormal basis for

$$\mathrm{span}\{1, \sin \pi x, \cos \pi x\}.$$

   Solution: Let $\psi_1(x) = 1$, $\psi_2(x) = \sin \pi x$ and $\psi_3(x) = \cos \pi x$.

   The first step in Gram-Schmidt is to take $\phi_1 = \psi_1/\|\psi_1\|$ and so we need to compute

$$\|\psi_1\| = \sqrt{\langle \psi_1, \psi_1\rangle} = \sqrt{\int_0^1 1^2\,dx} = \sqrt{1} = 1.$$

   So $\phi_1(x) = 1$.

   For ϕ_2 we compute $\psi_2 - \langle \psi_2, \phi_1\rangle$ and normalise. But

$$\langle \psi_2, \phi_1\rangle = \int_0^1 \psi_2(x)\overline{\phi_1(x)}\,dx = \int_0^1 \sin \pi x\,dx = \left[\frac{-1}{\pi}\cos \pi x\right]_0^1 = \frac{-1}{\pi}\cos\pi + \frac{1}{\pi}\cos 0 = \frac{2}{\pi}$$

So ϕ_2 is got by normalising $\psi_2 - \langle \psi_2, \phi_1 \rangle = \sin \pi x - 2/\pi$ and

$$\begin{aligned}
\|\psi_2 - \langle \psi_2, \phi_1 \rangle\| &= \sqrt{\langle \psi_2 - \langle \psi_2, \phi_1 \rangle, \langle \psi_2 - \langle \psi_2, \phi_1 \rangle \rangle} \\
&= \sqrt{\int_0^1 (\sin \pi x - 2/\pi)^2 \, dx} \\
&= \sqrt{\sin^2 \pi x - (4/\pi) \sin \pi x + 4/\pi^2 \, dx} \\
&= \sqrt{(1/2)(1 - \cos 2\pi x) - (4/\pi) \sin \pi x + 4/\pi^2 \, dx} \\
&= \sqrt{\left[\frac{x}{2} - \frac{1}{4\pi x} \sin 2\pi x + \frac{4}{\pi^2} \cos \pi x + \frac{4x}{\pi^2} \right]_0^1} \\
&= \sqrt{\frac{1}{2} - 0 - \frac{4}{\pi^2} + \frac{4}{\pi^2} - \left(0 - 0 + \frac{4}{\pi^2} + 0 \right)} \\
&= \sqrt{\frac{1}{2} - \frac{4}{\pi^2}}
\end{aligned}$$

Thus

$$\phi_2(x) = \frac{\sin \pi x - 2/\pi}{\sqrt{\frac{1}{2} - \frac{4}{\pi^2}}}$$

For ϕ_3 we compute $\psi_3 - \langle \psi_3, \phi_2 \rangle \phi_2 - \langle \psi_3, \phi_1 \rangle \phi_1$ and normalise. Now

$$\langle \psi_3, 1 \rangle = \int_0^1 \cos \pi x \, dx = \left[\frac{1}{\pi} \sin \pi x \right]_0^1 = 0 - 0 = 0$$

and

$$\langle \psi_3, \sin \pi x \rangle = \int_0^1 \cos \pi x \sin \pi x \, dx = \int_0^1 \frac{1}{2} \sin 2\pi x \, dx = \left[\frac{-1}{4\pi} \cos 2\pi x \right]_0^1 = \frac{-1}{4\pi} - \frac{-1}{4\pi} = 0$$

Thus $\langle \psi_3, \phi_1 \rangle = 0$ and $\langle \psi_3, \phi_2 \rangle = 0$. So we need to normalize ψ_3 to get $\phi_3 = \psi_3 / \|\psi_3\|$.

$$\|\phi_3\|^2 = \int_0^1 \cos^2 \pi x \, dx = \int_0^1 \frac{1}{2}(1 + \cos 2\pi x) \, dx = \left[\frac{x}{2} + \frac{1}{4\pi} \sin 2\pi x \right]_0^1 = \frac{1}{2}.$$

Thus $\phi_3 = \sqrt{2} \cos \pi x$.

$\{\phi_1, \phi_2, \phi_3\} = \{1, \frac{\sin(\pi x) - 2/\pi}{\sqrt{\frac{1}{2} - \frac{4}{\pi^2}}}, \sqrt{2} \cos \pi x\}$ is the result of Gram-Schmidt.

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