Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS & SS Mathematics **Trinity Term 2015**
JS & SS TSM Mathematics

Module MA3422— Functional Analysis II

Day PLACE TIME

Prof. R. Timoney

Instructions to Candidates:

Credit will be given for the best 3 questions answered.

All questions have equal weight.

'Formulae & tables' are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used

In the questions $\mathbb{K}$ denotes one of $\mathbb{R}$ or $\mathbb{C}$.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) [6 points] Define the notions of an inner product space (over $\mathbb{K}$) and of a Hilbert space. Give a finite dimensional and an infinite dimensional example of a Hilbert space (without proofs).

Solution:

Definition 1.1. An *inner product space* (also known as a pre-Hilbert space) is a vector space V over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) together with a map

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{K}$$

satisfying (for $x, y, z \in V$ and $\lambda \in \mathbb{K}$):

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

(iv) $\langle x, x \rangle \geq 0$

(v) $\langle x, x \rangle = 0 \Rightarrow x = 0$

An inner product on V gives rise to a norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

If the inner product space is complete in this norm (or in other words, if it is complete in the metric arising from the norm, or if it is a Banach space with this norm) then we call it a *Hilbert space*.

Examples 1.2. (i) $\mathbb{C}^n$ with the inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ is a Hilbert space (over $\mathbb{K} = \mathbb{C}$). (Here we mean that $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$.)

(ii) ℓ^2 with the inner product

$$\langle a, b \rangle = \sum_{j=1}^{\infty} a_j \overline{b_j}$$

is a Hilbert space over $\mathbb{K}$ (where we mean that $a = \{a_j\}_{j=1}^{\infty}$, $b = \{b_j\}_{j=1}^{\infty}$).

(b) [7 points] Outline the steps required to show that a separable inner product space has an orthonormal basis.

Solution: Starting with a countable dense subset $S = \{x_1, x_2, \ldots\}$ in an inner product space V we define $\{\psi_1, \psi_2, \ldots\}$ by $\psi_1 = x_j$ for the smallest j with $x_j \neq 0$ and $\psi_{k+1} = x_j$ for the smallest j with $x_j \notin \operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_j\}$. Then $x_n \in \operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_n\}$, $S \subset \operatorname{span}\{\psi_1, \psi_2, \ldots\}$ and so $\operatorname{span}\{\psi_1, \psi_2, \ldots\}$ is dense in V.

Then apply Gram-Schmidt to find an orthonormal sequence $\phi_1, \phi_2, \ldots$ with $\operatorname{span}\{\phi_1, \ldots, \phi_n\} = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ for each n.

Being orthonormal with dense linear span $\phi_1, \phi_2, \ldots$ is an orthonormal basis for V (as can be proved by using the fact that if $x \in V$, then $\sum_{j=1}^n \langle x, \phi_j \rangle \phi_j$ is the closest element in $\operatorname{span}\{\phi_1, \ldots, \phi_n\}$ to x).

(c) [7 points] State the Riesz representation theorem for separable Hilbert spaces and outline the steps required for a proof of it.

Solution:

Theorem 1.3 (Riesz representation theorem)**.** *Let H be a (separable) Hilbert space and $\alpha \in H^*$. Then there exists $y \in H$ such that*

$$\alpha(x) = \langle x, y \rangle \qquad \text{for all } x \in H.$$

Conversely, given $y \in H$, $\alpha_y(x) = \langle x, y \rangle$ defines an element of H^* and moreover $\|\alpha_y\| = \|y\|$.

For the proof, the case of a finite dimensional H is quite easy. Take an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$ for H and define y by

$$y = \sum_{j=1}^n \overline{\alpha(\phi_j)} \phi_j$$

In the case of ℓ^2 we do something similar using the standard basis $e_1, e_2, \ldots$

$$y = \sum_{n=1}^\infty \overline{\alpha(\phi_n)} \phi_n$$

but we have to establish that this defines an element of ℓ^2. The idea there is that $\sum_{j=1}^n |\alpha(\phi_j)|^2 \leq \|\alpha\|^2$. There is a little more argument to show $\alpha(x) = \langle x, y \rangle$ for general x (using convergence).

The case of a general separable infinite dimensional H can be reduced to ℓ^2 since there is an isometric linear isomorphism $T\colon H \to \ell^2$.

The converse part is a consequence of Cauchy-Schwarz and works the same way for all H. Cauchy-Schwarz implies $\|\alpha_y\| \leq \|y\|$ and then (if $y \neq 0$) considering the vector $x = y/\|y\|$ shows $\|\alpha_y\| \geq \|y\|$.

2. (a) [6 points] State the closed graph theorem. What is the other major result used in its proof?

Solution:

Theorem 2.1 (Closed graph theorem). *If E, F are Banach spaces and $T\colon E \to F$ is a linear transformation, then T is bounded if and only if its 'graph'*

$$\{(x, y) \in E \times F : y = Tx\}$$

is a closed subset of $E \times F$ in the product topology.

The proof uses the open mapping theorem, the consequence that bijective bounded linear operators between Banach spaces have bounded inverses.

(b) [7 points] State the Hahn-Banach theorem and use it to show that there is a continuous linear functional $\alpha\colon C[0,1] \to \mathbb{K}$ with $\alpha(f) = 0$ for $f(x) \equiv 1$ and $\alpha(g) = 1$ for $g(x) \equiv x$.

Solution:

Theorem 2.2 (Hahn-Banach Theorem for $\mathbb{K}$). *Let E be a vector space over $\mathbb{K}$ and M a vector subspace. Suppose $p\colon E \to [0, \infty)$ is a seminorm on E and suppose*

$$\alpha\colon M \to \mathbb{K}$$

is a linear functional satisfying

$$|\alpha(x)| \leq p(x) \qquad \text{for all } x \in M.$$

Then there exists an extension $\beta\colon E \to \mathbb{K}$ of α which is linear and satisfies

(i) $\beta(x) = \alpha(x)$ for all $x \in M$ (i.e. β extends α)

(ii) $|\beta(x)| \leq p(x)$ for all $x \in E$.

To use the theorem as requested, let $M = \operatorname{span}\{1, x\} = \operatorname{span}\{f, g\}$. Define $\alpha\colon M \to \mathbb{K}$ by $\alpha(\lambda f + \mu g) = \mu$. This is well defined since f and g are linearly independent: if $\lambda f + \mu g = 0$ then $\lambda f(0) + \mu g(0) = 0$, so $\lambda = 0$; and then also $\mu = 0$.

Since M is finite dimensional α is bounded on M and so there is $C > 0$ with $|\alpha(h)| \leq C\|h\|$ for $h \in M$. (Here $\|h\|$ is the norm of $C[0,1]$, the supnorm.) By applying Hahn-Banach with $p(x) = C\|x\|$ (a seminorm on $C[0,1]$) we know that α can be extended to a bounded linear functional on $C[0,1]$

(c) [7 points] If E is a normed space, define the caonical embedding $J\colon E \to E^{**}$ of E in its bidual and show that J is linear and isometric.

Solution: If E is any normed space, then there is a natural linear map

$$J\colon E \to E^{**} = (E^*)^*$$

given by

$$J(x) = \text{ point evaluation at } x.$$

In other words, for each $x \in E$

$$\begin{array}{rcl} J(x)\colon E^* & \to & \mathbb{K} \\ J(x)(\alpha) & = & \alpha(x). \end{array}$$

The map $J(x)\colon E^* \to \mathbb{K}$ is linear since

$$J(x)(\alpha + \lambda\beta) = (\alpha + \lambda\beta)(x) = \alpha(x) + \lambda\beta(x) = J(x)(\alpha) + \lambda J(x)(\beta)$$

and

$$|J(x)(\alpha)| = |\alpha(x)| \leq \|\alpha\| \, \|x\|$$

shows that $J(x)$ is bounded on E^*. Thus $J(x) \in (E^*)^*$ and, in fact,

$$\|J(x)\|_{(E^*)^*} \leq \|x\|.$$

By a corollary to Hahn-Banach, given $x \in E$, there exists $\alpha \in E^*$, with $\|\alpha\| = 1$ and $|\alpha(x)| = \|x\| = \|\alpha\| \, \|x\|$. This shows that $\|J(x)\| \geq \|x\|$. Therefore $\|J(x)\| = \|x\|$ and J is an isometry onto its range. It follows that J must be injective.

3. (a) [6 points] Give the definition of a locally convex topological vector space and show that if E is such a space, $U \subseteq E$ is open and $x_0 \in E$, then $x_0 + U$ must also be open.

Solution:

Definition 3.1. A *topological vector space* is a vector space E over $\mathbb{K}$ together with a topology $\mathcal{T}$ on E such that

(i) addition $E \times E \to E$ $((x, y) \mapsto x + y)$ is continuous (using the product topology on E and the given topology on E);

(ii) scalar multiplication $\mathbb{K} \times E \to E$ $((\lambda, x) \mapsto \lambda x)$ is continuous.

Let E, x_0 and U be as given.

Since $x \mapsto (-x_0, x) \colon E \to E \times E$ and $(x, y) \mapsto x + y \colon E \times E \to E$ are continuous, so is the composition $x \mapsto -x_0 + x$ continuous. The inverse image of U under this map is $\{x : -x_0 + x \in U\} = x_0 + U$ and so $x_0 + U$ is open.

(b) [7 points] If E is a normed space, define the weak topology on E and the weak*-topology on E^*.

Solution:

Definition 3.2. If E is normed space then the weak topology on E is the topology generated by the seminorms

$$p_\alpha(x) = |\alpha(x)| \qquad (\alpha \in E^*).$$

Definition 3.3. If E is normed space then the weak topology on the dual E^* is the topology generated by the seminorms

$$p_x(\alpha) = |\alpha(x)| \qquad (x \in E).$$

(c) [7 points] Show that if E is infinite dimensional and if $U \subseteq E$ is open for the weak-topology with $0 \in U$, then U contains a vector subspace of E of finite codimension in E.

Solution: As U is open in the weak topology and contains 0, there must be $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_n \in E^*$ and $r_1, r_2, \dots, r_n > 0$ so that

$$|\alpha_j(x)| < r_j \text{ for each } j \text{ with } 1 \leq j \leq n \Rightarrow x \in U$$

But then $x \in \bigcap_{j=1}^n \ker \alpha_j \Rightarrow x \in U$ and $\bigcap_{j=1}^n \ker \alpha_j$ is a vector subspace of E of codimension at most n.

Longer justification for the above rather terse explanation:

A base for the weak toplogogy consists of finite intersections $\bigcap_{j=1}^n U_j$ of sets U_j that are open for a seminorm $p_{\alpha_j}(x) = |\alpha_j(x)|$ (with $\alpha_j \in E^*$). So if $0 \in U$ with U open in the weak topology, there is such a basic open set $\bigcap_{j=1}^n U_j$ with $0 \in \bigcap_{j=1}^n U_j \subset U$.

As $0 \in U_j$ and U_j open with respect to the topology for the seminorm p_{α_j}, there is $r_j > 0$ with

$$B_{p_{\alpha_j}}(0, r_j) \subset U_j$$

(or such that $x \in E, |\alpha_j(x - 0)| < r_j \Rightarrow x \in U_j$). So

$$|\alpha_j(x)| < r_j \text{ for each } j \text{ with } 1 \leq j \leq n \Rightarrow x \in U$$

Even longer:

If $0 \in U$ and $U \subset E$ open for the weak toplogy, we know there is a seminorm $q \colon E \to [0, \infty$ on E that is continuous for the weak topology and has

$$B_q(0, 1) \subset U$$

Also q satisfies

$$q(x) \leq C_1|\alpha_1(x)| + \cdots + C_n|\alpha_n(x)|$$

for some $\alpha_1, \dots, \alpha_n \in E^*$ and constants $C_, \dots, C_n > 0$. Take $r_j = 1/(nC_j)$ and then

$$|\alpha_j(x)| < r_j \text{ for each } j \text{ with } 1 \leq j \leq n \Rightarrow q(x) < \sum_{j=1}^n C_j r_j = \sum_{j=1}^n \frac{1}{n} = 1 \Rightarrow x \in U$$

4. (a) [8 points] In $H = L^2[0,1]$ use the Gram-Schmidt process to find an orthonormal basis for

$$\mathrm{span}\{1, x\}.$$

Solution: Let $\psi_1(x) = 1$ and $\psi_2(x) = x$.

The first step of Gram-Schmidt is to take $\phi_1 = \psi_1/\|\psi_1\|$ and so we need to calculate

$$\|\psi_1\| = \sqrt{\int_0^1 |\psi_1(x)|^2\,dx} = \sqrt{\int_0^1 1\,dx} = 1$$

So $\psi_1(x) \equiv 1$.

The next step is to compute $\psi_2 - \langle \psi_2, \phi_1 \rangle \phi_1$ (and later normalise that).

$$\langle \psi_2, \phi_1 \rangle = \int_0^1 x\bar{1}\,dx = [x^2/2]_0^1 = 1/2$$

So $\psi_2 - \langle \psi_2, \phi_1 \rangle \phi_1$ is $x - 1/2$.

$$\begin{aligned} \|x - 1/2\|_2 &= \sqrt{\int_0^1 (x - 1/2)^2\,dx} \\ &= \sqrt{\int_0^1 x^2 - x + 1/4\,dx} \\ &= \sqrt{[x^3/3 - x^2/2 + x/4]_0^1} \\ &= \sqrt{1/3 - 1/2 + 1/4} = \sqrt{1/12} \end{aligned}$$

So $\phi_2(x) = \sqrt{12}(x - 1/2)$.

The orthonormal basis is $\{\phi_1, \phi_2\}$.

(b) [12 points] Let $E = \ell^2 \cap \ell^4$ with the norm $\|\cdot\|_4$ and $F = \ell^2$. Define $T \colon E \to F$ by $Tx = x$. Show that T is not bounded but has closed graph.

Solution: Consider $x_n = (1, 1, \ldots, 1, 0, 0, \ldots)$ with n ones. Then $\|x_n\|_4 = (1^4 + 1^4 + \cdots 1^4 + 0)^{1/4} = n^{1/4}$ while $\|Tx_n\| = \|x_n\|_2 = (1^2 + 1^2 + \cdots + 1^2 + 0 = n^{1/2}$. There is no finite constant C with $n^{1/2} \le Cn^{1/4}$ for all n and hence T is not bounded.

Suppose now (x_n, y_n) is in the graph of T for all n (different x_n from above) and $(x_n, y_n) \to (x, y)$ in $E \times F$ (as $n \to \infty$). So $y_n = Tx_n = x_n$. Also $x_n \to x$ in E

and $y_n = x_n \to y$ in F. So $\|x_n - x\|_4 \to 0$ as $n \to \infty$ and $\|x_n - y\|_2 \to 0$ as $n \to \infty$.

Then $x \in \ell^4$ and $y \in \ell^2$. But also, the k^{th} term of x_n converges (in $\mathbb{K}$) to the k^{th} term of x (for each $k \in \mathbb{N}$) and to the k^{th} term of y.

That is because if $x_{n,k}$ is the k^{th} term of x_n and $x_{\infty,k}$ is the k^{th} term of x, then $|x_{n,k} - x_{\infty,k}| \leq \|x_n - x\|_4$ and a similar fact holds for the norm $\| \cdot \|_2$.

So $x = y \in \ell^4 \cap \ell^2$. That means $(x, y) = (x, x)$ is in the graph of T and thus the graph is closed.