MA3422 2016–17

Chapter 3: Weak topologies and Tychonoff's theorem

Contents

3.1 Tychonoff's theorem

Theorem 3.1.1 (Tychonoff's theorem)**.** *Let $(X_\alpha)_{\alpha\in A}$ be a nonempty family of nonempty topological spaces X_α (for $\alpha \in A =$ some index set).*

Then $X = \prod_{\alpha\in A}$ is compact if and only if X_α is compact for each $\alpha \in A$.

Proof. $\Rightarrow$: (easy part) We denote by $\pi_\beta \colon X \to X_\beta$ the coordinate projection onto the β-coordinate (for $\beta \in A$). That is $\pi_\beta((x_\alpha)_{\alpha\in A}) = x_\beta$. Since the spaces X_α are nonempty, π_β is surjective. Also, by the way the product topology is defined (MA3421:1.5.10), π_β is continuous for each β. Thus $X_\beta = \pi_\beta(X)$ is compact (the continuous image of a compact space — Proposition MA3421:1.3.10).

$\Leftarrow$: (This is the real content of the theorem. As the proof relies on using Zorn's lemma in a way that is quite tricky, we relegate it to an appendix.)

$\square$

Example 3.1.2. Every generalised cube $\prod_{i\in I}[a_i, b_i]$ (where I is an index set and $a_i, b_i \in \mathbb{R}$, $a_i \le b_i \forall i \in I$) is compact.

Compactness follows from Tychonoff's theorem.

3.2 Locally convex topological vector spaces

Remark 3.2.1. Recall that given a seminorm $p\colon E \to [0,\infty)$ on a vector space E over $\mathbb{K}$, we can define an associated semi-metric $d_p\colon E \times E \to [0,\infty)$ by $d_p(x, y) = p(x - y)$. We can then define a topology on E by taking as a base for the topology the open balls

$$B_p(x, r) = \{y \in E : p(x - y) < r\}$$

for $r > 0$. We will denote this topology $\mathcal{T}_p$ and call it the topology defined by p.

Maybe we should recall what a seminorm is.

Definition 3.2.2. A *seminorm* on a vector space E over the field $\mathbb{K}$ is a function $p\colon E \to [0,\infty) \subseteq \mathbb{R}$ which satisfies the following properties

(i) (Triangle inequality) $p(x+y) \leq p(x) + p(y)$ (all $x, y \in E$);

(ii) (scaling property) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{K}$, $x \in E$;

(iii) $p(x) \geq 0$ for $x \in E$ (which is already given by $p \colon E \to [0, \infty)$).

A vector space E over $\mathbb{K}$ together with a chosen norm $\|\cdot\|$ is called a *normed space* (over $\mathbb{K}$) and we write $(E, \|\cdot\|)$.

Definition 3.2.3. If E is a vector space over $\mathbb{K}$ and $x, y \in E$, then a *convex combination* of x and y means an element of E of the form $(1-t)x + ty$ where $0 \leq t \leq 1$ ($t \in \mathbb{R}$).

A subset $S \subseteq E$ is called *convex* if S contains all convex combinations of pairs of points $x, y \in S$.

Example 3.2.4. Balls $B_p(x, r)$ with respect to a seminorm (or norm) p are convex.

Definition 3.2.5. A *topological vector space* is a vector space E over $\mathbb{K}$ together with a topology $\mathcal{T}$ on E such that

(i) addition $E \times E \to E$ ($(x, y) \mapsto x + y$) is continuous (using the product topology on E and the given topology on E);

(ii) scalar multiplication $\mathbb{K} \times E \to E$ ($(\lambda, x) \mapsto \lambda x$) is continuous.

A topological vector space is called *a locally convex topological vector space* if it also satisfies:

(iii) there is a base for $\mathcal{T}$ consisting of convex (open) sets.

Proposition 3.2.6. *Given a family $\mathcal{P}$ of seminorms on a vector space E over $\mathbb{K}$, there is a weakest topology $\mathcal{T}_{\mathcal{P}}$ on E which contains all the topologies $\mathcal{T}_p$ for $p \in \mathcal{P}$.*

Moreover $(E, \mathcal{T}_{\mathcal{P}})$ is a topological vector space and a base for the topology $\mathcal{T}_{\mathcal{P}}$ can be given by the balls $B_q(x, r)$ with $x \in E$, $r > 0$ and q a seminorm on E continuous with respect to $\mathcal{T}_{\mathcal{P}}$.

A seminorm q on E is continuous with respect to $\mathcal{T}_{\mathcal{P}}$ if and only if there are $n \geq 1, p_1, p_2, \ldots, p_n \in \mathcal{P}$ and positive scalars $C_1, C_2, \ldots, C_n$ satisfying

$$q(x) \leq \sum_{j=1}^{n} C_j p_j(x) \quad (x \in E).$$

Proof. By Theorem MA3421:1.5.3 there is a weakest topology $\mathcal{T}_{\mathcal{P}}$ containing the union $\bigcup_{p \in \mathcal{P}} \mathcal{T}_p$. Moreover, since each set in $\mathcal{T}_p$ is a union of balls $B_p(x, r)$, the topology $\mathcal{T}_{\mathcal{P}}$ is the weakest on that contains

$$\{B_p(x, r) : x \in X, r > 0, p \in \mathcal{P}\}$$

Moreover, from the constructive proof of Theorem MA3421:1.5.3, there is a base for the topology $\mathcal{T}_{\mathcal{P}}$ consisting of finite intersection of balls $B_p(x, r)$.

If q is a seminorm on E continuous with respect to $\mathcal{T}_{\mathcal{P}}$, then $B_q(0,1) = \{x \in E : q(x) < 1\}$ must be open with respect to $\mathcal{T}_{\mathcal{P}}$ (as the inverse image of an open set) and so there are finitely many balls $B_{p_j}(x_j, r_j)$ such that

$$0 \in \bigcap_{j=1}^{n} B_{p_j}(x_j, r_j) \subseteq B_q(0,1)$$

Replacing r_j by $r'_j = r_j - p(x_j)$ we have $0 \in B_{p_j}(0, r'_j) \subseteq B_{p_j}(x_j, r_j)$ and so

$$0 \in \bigcap_{j=1}^{n} B_{p_j}(0, r'_j) \subseteq B_q(0,1).$$

Now if $C_j = 1/r'_j$, then $\sum_{j=1}^{n} C_j p_j(x) < 1$ implies $p_j(x) < 1/C_j = r'_j$ for $1 \leq j \leq n$, hence $x \in B_{p_j}(0, r'_j)$ ($1 \leq j \leq n$) so that $q(x) < 1$. It follows by scaling x that

$$q(x) \leq \sum_{j=1}^{n} C_j p_j(x) \tag{1}$$

holds for all $x \in E$.

Thus we have shown that each continuous seminorm satisfies the inequality (for some n, p_j and C_j). Conversely, suppose q is a seminorm on E that satisfies (1). The argument just given shows that

$$0 \in \bigcap_{j=1}^{n} B_{p_j}(0, r/C_j) \subseteq B_q(0,r)$$

(so that the origin is in the $\mathcal{T}_{\mathcal{P}}$ interior of $B_q(0,r)$) and by translation we get that,for any $x \in E$

$$x \in x + \bigcap_{j=1}^{n} B_{p_j}(0, r/C_j) = \bigcap_{j=1}^{n} B_{p_j}(x, r/C_j) \subseteq x + B_q(0,r) = B_q(x,r)$$

(so that the centre is in the $\mathcal{T}_{\mathcal{P}}$ interior of $B_q(x,r)$). It follows quite easily that $\mathcal{T}_q$ open sets are open in $\mathcal{T}_{\mathcal{P}}$. In particular balls $B_q(x,r)$ (with respect to continuous seminorms q, or more generally seminorms q that satisfy (1)) are open for $(E, \mathcal{T}_{\mathcal{P}})$.

We did also claim that seminorms q satisfying (1) must be continuous. But we have just shown that, for such q, $\mathcal{T}_q \subseteq \mathcal{T}_{\mathcal{P}}$. Then since q is $\mathcal{T}_q$-continuous (exercise), q must be $\mathcal{T}_{\mathcal{P}}$-continuous.

To show that the balls $B_q(x,r)$ form a base for the topology $\mathcal{T}_{\mathcal{P}}$ suppose $x \in U$ with $U \in \mathcal{T}_{\mathcal{P}}$. Then there are $n, p_1, p_2, \ldots, p_n \in \mathcal{P}$ and positive $r_1, r_2, \ldots, r_n$ so that $x \in \bigcap_{j=1}^{n} B_{p_j}(x_j, r_j) \subseteq U$ (because finite intersection of balls $B_p(x,r)$ form a base). Let $r'_j = r_j - p_j(x - x_j)$ and then we have

$$x \in \bigcap_{j=1}^{n} B_{p_j}(x, r'_j) \subseteq U.$$

Let $q = \sum_{j=1}^n (1/r'_j) p_j$, which is then a continuous seminorm on $(E, \mathcal{T}_{\mathcal{P}})$ (as it satisfies (1)). Then $q(y-x) < 1$ implies $p_j(y-x) < r'_j$ for $1 \leq j \leq n$, and so

$$x \in B_q(x, 1) \subseteq \bigcap_{j=1}^n B_{p_j}(x, r'_j) \subseteq U.$$

It remains to show that addition and multiplication are continuous. Let $U \subseteq E$ be open in $\mathcal{T}_{\mathcal{P}}$. To show that the inverse image of U under addition is open, pick $(x_0, y_0) \in E \times E$ with $x_0 + y_0 \in U$. We know there is a continuous seminorm q with $x_0 + y_0 \in B_q(x_0 + y_0, 1) \subseteq U$.

The argument now is really that addition is continuous in $\mathcal{T}_q$. In detail, if $(x, y) \in B_q(x_0, 1/2) \times B_q(y_0, 1/2)$ then

$$q((x+y)-(x_0+y_0)) = q((x-x_0)+(y-y_0)) \leq q(x-x_0)+q(y-y_0) < 1 \Rightarrow x+y \in B_q(x_0+y_0, 1).$$

Thus the open set $B_q(x_0, 1/2) \times B_q(y_0, 1/2)$ (in the product topology on$E \times E$) is in the inverse image of U.

If $(\lambda_0, x_0) \in \mathbb{K} \times \mathbb{R}$ has $\lambda_0 x_0 \in U$, again pick q with $B_q(\lambda_0 x_0, 1) \subseteq U$ and the argument is that scalar multiplication is continuous in $\mathcal{T}_q$.

If $|\lambda - \lambda_0| < \min(1/(2(q(x_0)+1)), |\lambda_0| + 1)$ and $q(x - x_0) < 1/(4|\lambda_0| + 2)$ then

$$\begin{aligned}
q(\lambda x - \lambda_0 x_0) &\leq q(\lambda x - \lambda x_0) + q(\lambda x_0 - \lambda_0 x_0) \\
&= |\lambda| q(x - x_0) + |\lambda - \lambda_0| q(x_0) \\
&\leq (|\lambda_0| + |\lambda - \lambda_0|) \frac{1}{4(|\lambda_0| + 2)} + \frac{q(x_0)}{2(q(x_0)+1)} \\
&\leq \frac{2|\lambda_0| + 1}{4|\lambda_0| + 2} + \frac{1}{2} \\
&< 1
\end{aligned}$$

Since the balls $B_q(x, r)$ are convex (and form a base), we have verified that $(E, \mathcal{T}_{\mathcal{P}})$ is a locally convex topological vector space. $\square$

Example 3.2.7. Let $G \subseteq \mathbb{C}$ be an open set and let $H(G)$ denote the space of all holomorphic functions $f \colon G \to \mathbb{C}$ (also called analytic functions). For $K \subset G$ compact introduce a seminorm $\|\cdot\|_K$ on $H(G)$ by $\|f\|_K = \sup_{z \in K} |f(z)|$.

Then the topology on $H(G)$ generated by the seminorms $\|\cdot\|_K$ is a locally convex topological vector space topology. It is the usual topology used on $H(G)$, often called the *compact open topology*.

Proposition 3.2.8. *Let E be a vector space over $\mathbb{K}$, $\mathcal{P}$ a family of seminorms on E and $\mathcal{T}_{\mathcal{P}}$ the topology defined by $\mathcal{P}$. If $f\colon X \to E$ is a function from a topological space X into E, then f is continuous for $\mathcal{T}_{\mathcal{P}}$ if and only if it is continuous for $\mathcal{T}_p$ for each $p \in \mathcal{P}$.*

Proof. $\Rightarrow$: The identity map $(E, \mathcal{T}_{\mathcal{P}}) \to (E, \mathcal{T}_p)$ is continuous for each $p \in \mathcal{P}$. So if f is continuous from X into $(E, \mathcal{T}_{\mathcal{P}})$, then it is continuous from X into $(E, \mathcal{T}_p)$ (for each $p \in \mathcal{P}$).

$\Leftarrow$: Suppose f is continuous from X into $(E, \mathcal{T}_p)$ (for each $p \in \mathcal{P}$) and consider a basic open set for $\mathcal{T}_{\mathcal{P}}$ of the form

$$B = \bigcap_{j=1}^{n} B_{p_j}(x_j, r_j)$$

where $p_j \in \mathcal{P}$ and $r_j > 0$ (each j). Then the open ball $B_{p_j}(x_j, r_j)$ is in $\mathcal{T}_{p_j}$ and so $f^{-1}(B_{p_j}(x_j, r_j))$ must be open in X. Then so is

$$f^{-1}(B) = \bigcap_{j=1}^{n} f^{-1}(B_{p_j}(x_j, r_j))$$

□

Remark 3.2.9. Using Proposition 3.2.8 (which you can see is not so difficult to show) we can give a simpler proof of the part of Proposition 3.2.6 that states that $\mathcal{T}_{\mathcal{P}}$ is a locally convex topological vector space topology.

The locally convex part is OK since a base for the topology $\mathcal{T}_{\mathcal{P}}$ consists of (convex) finite intersections

$$B = \bigcap_{j=1}^{n} B_{p_j}(x_j, r_j)$$

where $p_j \in \mathcal{P}$ and $r_j > 0$ (each j).

By Proposition 3.2.8, to show that addition $: E \times E \to E$ is continuous from the product $\mathcal{T}_{\mathcal{P}}$ topology to the $\mathcal{T}_{\mathcal{P}}$ topology, we just need to show it is continuous into $\mathcal{T}_p$ for each $p \in \mathcal{P}$.

But that follows from just the last part of the proof of Proposition 3.2.6 (the product $\mathcal{T}_p$ topology on $E \times E$ is weaker than the product $\mathcal{T}_{\mathcal{P}}$ topology).

Similarly one can show that multiplication by scalars is continuous.

3.3 Weak and weak*-topologies

Definition 3.3.1. If E is normed space then the weak topology on E is the topology generated by the seminorms

$$p_\alpha(x) = |\alpha(x)| \qquad (\alpha \in E^*).$$

Definition 3.3.2. If E is normed space then the weak* topology on the dual E^* is the topology generated by the seminorms

$$p_x(\alpha) = |\alpha(x)| \qquad (x \in E).$$

Proposition 3.3.3. *The weak topology on a normed space E is weaker than the norm topology, and strictly weaker unless E is finite dimensional.*

The weak-topology on E^* is weaker than the norm topology, and strictly weaker unless E is finite dimensional.*

Proof. The identity linear map from $(E, \|\cdot\|)$ to E with any one of the seminorms $p_\alpha(\cdot) = |\alpha(\cdot)|$ is continuous because it satisfies $p_\alpha(x) \leq \|\alpha\|\|x\|$. It follows that each topology $\mathcal{T}_{p_\alpha}$ is weaker than the norm topology and so therefore is the weak topology weaker than the norm topology.

If the weak topology was the same as the norm topology, then $q(x) = \|x\|$ would be a continuous (semi)norm on E for the weak topology, and so Proposition 3.2.6 assure us that there must be $\alpha_1, \alpha_2, \ldots, \alpha_n \in E^*$ and constants $C_j > 0$ so that

$$q(x) = \|x\| \leq \sum_{j=1}^{n} C_j p_{\alpha_j}(x) = \sum_{j=1}^{n} C_j |\alpha_j(x)|$$

holds for all $x \in E$. It follows that if $x \in \bigcap_{j=1}^{n} \ker \alpha_j$, then $\|x\| = 0 \Rightarrow x = 0$.

However $\bigcap_{j=1}^{n} \ker \alpha_j$ has codimension at most n in E. Thus if E is infinite dimensional, then we have to have $\bigcap_{j=1}^{n} \ker \alpha_j \neq \{0\}$ and we have a contradiction.

> If you are not happy with the codimension idea, here is a rather elementary argument. If E is infinite dimensional, then there must be $n+1$ linearly independent elements $x_1, x_2, \ldots, x_{n+1} \in E$. The condition for an element $x = \sum_{k=1}^{n+1} t_k x_k$ (in their span) to be in $\bigcap_{j=1}^{n} \ker \alpha_j$ is
>
> $$\sum_{k=1}^{n+1} t_k \alpha_j(x_k) = 0 \quad (1 \leq j \leq n).$$
>
> That is n homogeneous linear equations in $n+1$ unknowns $t_1, t_2, \ldots, t_{n+1}$ and there is bound to be a nonzero solution.

The proof for the weak*-topology is really almost identical. In a way the roles of x and α have to be exchanged and that is more or less it.

We do need to know that if E is infinite dimensional, then so is E^*. One way to see that is to notice that if $\dim E^* < \infty$, then its dual $(E^*)^* = E^{**}$ would have the same finite dimension. Since there is a canonical (linear isometric) embedding $J \colon E \to E^{**}$, we'd have to have $\dim E < \infty$ also. □

Notation 3.3.4. For a normed space E, we use B_E for the open unit ball of E, $B_E = \{x \in E : \|x\| < 1\}$, and $\bar{B}_E = \{x \in E : \|x\| \leq 1\}$ for the closed unit ball.

Theorem 3.3.5 (Banach-Alaoglu)**.** *Let E be a normed space. Then the closed unit ball $\bar{B}_{E^*}$ is compact in the weak*-topology.*

Proof. For each $x \in E$, let $D_x = \{\lambda \in \mathbb{K} : |\lambda| \leq \|x\|\}$ and define a map $\tau \colon \bar{B}_{E^*} \to \prod_{x \in E} D_x$ by

$$\tau(\alpha) = (\alpha(x))_{x \in E}$$

We claim that τ is a homeomorphism from $(\bar{B}_{E^*}, \mathcal{T}_{w^*})$ onto its range and that the range is compact. To show compactness of the range, we show it is closed in the product space $\prod_{x \in E} D_x$ (which is compact by Tychonoff's theorem).

The first point is that τ is properly defined (has values in the product $\prod_{x \in E} D_x$ because $|\alpha(x)| \leq \|\alpha\|\|x\| \leq \|x\|$ (since $\|\alpha\| \leq 1$)).

The range of τ should be considered to be equipped with the subspace topology of the product space. If we write $X = \prod_{x \in E} D_x$ and $Y = \tau(\bar{B}_{E^*})$, then that means that the open sets for the topology on Y are the intersections $U \cap Y$ with Y of open sets $U \subseteq X$, and then we can say that these are unions of sets of the form $B \cap Y$ with B a basic open set of X. Similarly a basic open set for $(\bar{B}_{E^*}, \mathcal{T}_{w^*})$ (weak*-topology) would be the intersection of a basic open set of the weak*-topology of E^* with the ball $\bar{B}_{E^*}$, hence of the form

$$\bar{B}_{E^*} \cap \bigcap_{j=1}^{n} \{\alpha \in E^* : p_{x_j}(\alpha - \alpha_j) < r_j\} = \bar{B}_{E^*} \cap \bigcap_{j=1}^{n} \{\alpha \in E^* : |\alpha(x_j) - \alpha_j(x_j)| < r_j\}$$

(where $n \geq 0$, $x_j \in E$ and $\alpha_j \in E^*$ for $1 \leq j \leq n$).

To show that τ is continuous with values in the product space, it is enough to show that each coordinate function $\alpha \mapsto \alpha(x)$ is continuous (that is for each $x \in E$). So fix one $x \in E$ and call this map τ_x (so $\tau_x(\alpha) = \alpha(x)$). If $x = 0$, then we are looking at the constant map $\alpha \mapsto 0$ (which is continuous). If $x \neq 0$ consider the inverse image under τ_x of a disk $\{\lambda \in \mathbb{K} : |\lambda - \lambda_0| < r\}$ (or actually an interval if $\mathbb{K} = \mathbb{R}$), where $\lambda_0 \in \mathbb{K}$ and $r > 0$. By Hahn-Banach there is $\alpha_0 \in E^*$ with $\alpha_0(x) = \lambda_0$ and then the pullback under τ_x of the disk is

$$\tau_x^{-1}(\{\lambda \in \mathbb{K} : |\lambda - \lambda_0| < r\}) = \bar{B}_{E^*} \cap \{\alpha \in E^* : |\alpha(x) - \alpha_0(x)| < r\}$$

That's a basic open set in $(\bar{B}_{E^*}, \mathcal{T}_{w^*})$, hence open. So τ_x must be continuous (for all $x \in E$). So τ is continuous (by Theorem MA3421:1.5.13 (i)).

It is rather clear that τ is injective because $\tau(\alpha) = \tau(\beta)$ tells us that $\alpha(x) = \beta(x)$ for each $x \in E$, which is just that $\alpha = \beta$.

We can also see that τ is a homeomorphism onto its range in a way that is rather direct. Basic open sets in a product are described in Remarks MA3421:1.5.12. If $\pi_y \colon \prod_{x \in E} D_x \to D_y$ is the projection onto the y coordinate, then we get basic open sets by taking finite intersections of sets $\pi_y^{-1}(U)$ with $U \subset D_y$ open. But those U are unions of intersections of disks $B(\lambda_0, r)$ with D_y. So we can just consider finite intersections of sets

$$\pi_y^{-1}(B(\lambda_0, r) \cap D_y).$$

Then we take the intersections of these with the range of τ to get a subbase (or base if we take all finite intersections of these sets) for the topology on the range of τ inside the product.

Now

$$\tau^{-1}\left(\pi_y^{-1}(B(\lambda_0, r) \cap D_y)\right) = \tau_y^{-1}(B(\lambda_0, r) \cap D_y)$$

and we've just seen that we can write this set as

$$\bar{B}_{E^*} \cap \{\alpha \in E^* : |\alpha(y) - \alpha_0(y)| < r\} \tag{2}$$

(for some $\alpha_0 \in E^*$) as long as $y \neq 0$. The case $y = 0$ is not really a problem because $D_0 = \{0\}$ and so we either have the whole space or the empty set, and so we can restrict our attention to $y \neq 0$. Basic open sets in the weak*-topology of $\bar{B}_{E^*}$ can be taken to be finite intersections of exactly the sets (2). Since τ is a bijection onto its range that maps subbasic open sets (2) to subbasic open sets

$$\pi_y^{-1}(B(\lambda_0, r) \cap D_y) \cap \tau(\bar{B}_{E^*}),$$

it must be a homeomorphism.

It remains to show that the range of τ is closed in the product $\prod_{x \in E} D_x$ (which we know is compact by Tychonoff's theorem — Theorem 3.1.1).

We can view elements of $\prod_{x \in E} D_x$ as functions $\alpha \colon E \to \bigcup_{x \in E} D_x = \mathbb{K}$. The range of τ consists of those functions which are linear ($\alpha(\lambda x + y) = \lambda\alpha(x) + \alpha(y)$ for each $x, y \in E$ and $\lambda \in \mathbb{K}$) and also bounded by 1 ($|\alpha(x)| \leq \|x\|$ for each $x \in E$). To show that the linearity restriction gives a closed subset of $\prod_{x \in E} D_x$, consider for a moment fixed $y, z \in E$ and $\lambda \in \mathbb{K}$ and the map

$$\begin{aligned} : \prod_{x \in E} D_x &\to \mathbb{K}^3 \\ \alpha = (\alpha(x))_{x \in E} &\mapsto (\alpha(y), \alpha(z), \alpha(\lambda y + z)). \end{aligned}$$

That is a continuous map (each coordinate is continuous). The restriction $\alpha(\lambda y + z) = \lambda\alpha(y) + \alpha(z)$ on α defines a closed subset of $\prod_{x \in E} D_x$ because it is the inverse image of the closed set

$$\{(t_1, t_2, t_3) \in \mathbb{K}^3 : \lambda t_1 + t_2 - t_3 = 0\}.$$

$\left\{\alpha \in \prod_{x \in E} D_x : \alpha \text{ linear}\right\}$ is then the intersection over all $y, z \in E$ and $\lambda \in \mathbb{K}$ of these closed sets, and so it is a closed subset of $\prod_{x \in E} D_x$. Boundedness is automatic for linear α (because $\alpha(x) \in D_x \Rightarrow |\alpha(x)| \leq \|x\|$).

Thus the image of τ is closed and so compact. □

Theorem 3.3.6 (Goldstine)**.** *Let E be a normed space and $J \colon E \to E^{**}$ the canonical isometric embedding.*

Then the weak-closure of $J(B_E)$ (in the weak*-topology of $E^{**} = (E^*)^*$) is $\bar{B}_{E^{**}}$.*

Proof. Skipped for lack of time. □

Corollary 3.3.7. *Let E be a normed space. Then E is reflexive if and only if $\bar{B}_E$ is compact in the the weak topology.*

Proof. If E is reflexive, the weak topology on E coincides with the weak*-topology on E^{**}.

More precisely, the canonical embedding $J \colon E \to E^{**}$ (which is bijective in the case when E is reflexive) is a homeomorphism from the weak topology of E to the weak*-topology of

$E^{**} = (E^*)^*$. That is because the weak-topology on E is given by the seminorms $p_\alpha\colon E \to [0, \infty)$ with $p_\alpha(x) = |\alpha(x)|$ with $\alpha \in E^*$ and the weak*-topology on $(E^*)^*$ is also given by seminorms given by $\alpha \in E^*$. To make the notation clear we might denote the seminorms on E^{**} by $\tilde{p}_\alpha$ and write $\tilde{p}_\alpha(z) = |z(\alpha)|$ for $z \in E^{**}$. Now for $x \in E$ (and $z = J(x) \in E^{**}$) we have $\tilde{p}_\alpha(J(x)) = |J(x)(\alpha)| = |\alpha(x)| = p_\alpha(x)$, from which it can be deduced that J is a homeomorphism from the weak topology of E to the weak*-topology of E^{**}.

So if E is reflexive then because $J(\bar{B}_E) = \bar{B}_{E^{**}}$ is weak*-compact and J is a homeomorphism, it follows that $\bar{B}_E$ is weakly compact.

On the other hand if $\bar{B}_E$ is weakly compact, then so is $J(\bar{B}_E)$ weak*-compact and so closed in the weak*-topology of E^{**}. Hence by Goldstine's theorem $J(\bar{B}_E) = \bar{B}_{E^{**}}$ and it follows then by scaling that $J(E) = E^{**}$.

□

Proposition 3.3.8. *Let E be a normed space and F a vector subspace. Then the weak topology of F is the same as the relative weak topology (of F as a subset of E with the weak topology).*

Proof. The weak topology of F is given by the seminorms p_β for $\beta \in F^*$ with $p_\beta(y) = |\beta(y)|$ $(y \in F)$.

The weak topology of E is given by the seminorms p_α for $\alpha \in E^*$ with $p_\alpha(x) = |\alpha(x)|$ $(x \in E)$.

It can be checked that the relative weak topology of F as a subset of E with the weak topology is the topology on F given by the seminorms $p_\alpha|_F$, the restrictions to F of the p_α with $\alpha \in E^*$. Notice that $p_\alpha|_F = p_{\alpha|_F}$ But via the Hahn-Banach theorem every $\beta \in F^*$ is of the form $\beta = \alpha|F$ for some $\alpha \in E^*$ and it is a simple fact that $\alpha \in E^* \Rightarrow \alpha_F \in F^*$. So the seminorms $p_\alpha|_F$ $(\alpha \in E^*)$ are exactly the same as the seminorms p_β $(\beta \in F^*)$.

Hence the result. □

Proposition 3.3.9. *Let E be a reflexive Banach space and F a closed subspace. Then F is reflexive.*

Proof. Skipped.

(The idea is not hard. Show that F is closed in the weak topology for E, from which it follows that $\bar{B}_F = F \cap \bar{B}_E$ is closed in $\bar{B}_E$ (for the relative weak topology).

Compactness of $\bar{B}_E$ in the weak topology (of E) then implies compactness of $\bar{B}_F$ in the relative weak topology on F, which is the same as the weak topology of F (by Proposition 3.3.8).

Corollary 3.3.7 now implies that F is reflexive.) □

A Proof of Tychonoff's theorem

Theorem A.0.1 (Tychonoff's theorem 3.1.1)**.** *Let $(X_\alpha)_{\alpha\in A}$ be a nonempty family of nonempty topological spaces X_α (for $\alpha \in A =$ some index set).*

Proof. We proved the easy part $\Rightarrow$ and new er proxe $\Leftarrow$.

The proof presented here is quite tricky, but has the merit that it does not require a whole lot of further background used in some other proofs. This proof is from a book by L. Nachbin, 'The Haar Integral', Van Nostrand, 1965 (pages 102–103). The book is in the Hamilton Library 515.43 K5. The proof uses Zorn's lemma.)

To make the proof easier to write, we use some special notation.

- For $V_\alpha \subseteq X_\alpha$ a subset, we will use V_α^{-1} to denote $(\pi_\alpha)^{-1}(V_\alpha)$.
- We will use U to denote *open covers* of X.
- We will use V for open subsets of X and V_α for open subsets of X_α.

Suppose X is not compact. Then there is at least one open cover U of X with no finite subcover. Let $\mathcal{U}$ denote the collection of all such open covers U of X with no finite subcover.

We now use Zorn's lemma to show that there is a maximal member in $\mathcal{U}$ (maximal with respect to $\subseteq$; one cover U can be contained in another U' when U' has all the same open sets as U together with some more).

Here are the details of applying Zorn's lemma to $\mathcal{U}$. We need to establish that each chain $\mathcal{C}$ in $\mathcal{U}$ has an upper bound in $\mathcal{U}$. Notice first that $\mathcal{U} \neq \emptyset$ by our assumption that there is one $U \in \mathcal{U}$.

If $\mathcal{C}$ is not empty we consider $U' = \bigcup \mathcal{C} = \bigcup_{U\in\mathcal{C}} U$. Then U' is a collection of open subsets of X and U' covers X because if we take any $U \in \mathcal{C}$, then $X = \bigcup U = \bigcup_{V\in U} V \subseteq \bigcup U'$. We claim that $U' \in \mathcal{U}$ — in other words that U' has no finite subcover. To see this, suppose $\{V_1, V_2, \ldots, V_n\} \subset U'$ is a finite subcover. Then for each $1 \leq i \leq n$, there is $U^{(i)} \in \mathcal{C}$ so that $V_i \in U^{(i)}$. We can then use the fact that $\mathcal{C}$ is linearly ordered to replace $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ by the largest one of them. (For example, we must have either $U^{(1)} \subseteq U^{(2)}$ or $U^{(2)} \subseteq U^{(1)}$ and we can replace both by whichever is larger, to get V_1, V_2 contained in a common $U^{(2')} \in \mathcal{C}$.) If we denote the largest by $U^{(n')}$ we have a finite subcover $\{V_1, V_2, \ldots, V_n\} \subset U^{(n')} \in \mathcal{C} \subseteq \mathcal{U}$ — contradicting the definition of $U^{(n')} \in \mathcal{U}$. Hence $U' \in \mathcal{U}$. Also U' is clearly an upper bound for $\mathcal{C}$ ($U \subseteq U' \forall U \in \mathcal{C}$).

By Zorn's lemma then, there is a maximal $U \in \mathcal{U}$ — an open cover of X with no finite subcover, but as soon as we add one more open set V to U, the resulting $U \cup \{V\}$ will have a finite subcover (that is, if $V \notin U$).

Now fix $\alpha \in A$ for a moment and consider the collection of all open sets $V_\alpha \subseteq X_\alpha$ such that $V_\alpha^{-1} \in U$. We claim that this collection cannot cover X_α. If it did cover X_α, it would be an open cover of the compact X_α. So it would have a finite subcover $X_\alpha = V_\alpha^1 \cup V_\alpha^2 \cup \cdots \cup V_\alpha^n$. But then $X = (V_\alpha^1)^{-1} \cup (V_\alpha^2)^{-1} \cup \cdots \cup (V_\alpha^n)^{-1}$, where each $(V_\alpha^i)^{-1} \in U$. So U would have a finite subcover, which it does not.

As the open $V_\alpha \subseteq X_\alpha$ such that $V_\alpha^{-1} \in U$ do not cover X_α, we can select a point

$$x_\alpha \in X_\alpha \setminus \bigcup \{V_\alpha : V_\alpha \subseteq X_\alpha \text{ open}, V_\alpha^{-1} \in U\}$$

Do this for each $\alpha \in A$ and then we have a point $x = (x_\alpha)_{\alpha\in A} \in X$.

As U is a cover of X, there must be $V \in U$ with $x \in V$. Using the standard way to find a basis for the product topology on $X = \prod_{\alpha\in A} X_\alpha$, we know there are finitely many (distinct) $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, open sets $V_{\alpha_i} \subseteq X_{\alpha_i}$ ($1 \leq i \leq n$) so that

$$x \in \bigcap_{i=1}^{n} V_{\alpha_i}^{-1} \subset V.$$

For each i, $x_{\alpha_i} \in V_{\alpha_i}$ and by the choice of x_{α_i} we have $V_{\alpha_i}^{-1} \notin U$. By maximality of $U \in \mathcal{U}$, we know $U \cup \{V_{\alpha_i}^{-1}\}$ has a finite subcover. That is there exist

$$V^{i,j} \in U \quad (1 \leq j \leq n_i)$$

so that

$$X = V_{\alpha_i}^{-1} \cup \bigcup_{j=1}^{n_i} V^{i,j}.$$

It follows that

$$\bigcup_{i=1}^{n}\bigcup_{j=1}^{n_i} V^{i,j} \supseteq X \setminus \bigcap_{i=1}^{n} V_{\alpha_i}^{-1} \supseteq X \setminus V.$$

Thus V together with the sets $V^{i,j}$ ($1 \leq i \leq n$, $1 \leq j \leq n_i$) form a finite cover of X, and all these sets are in U. Hence U has a finite subcover — a contradiction.

Thus X must be compact. □

March 22: Remove some extra)'s and a typo in the proof of Theorem 3.3.5.

Richard M. Timoney (March 22, 2017)