MA3421 (Functional Analysis 1) Tutorial sheet 9

[December 8, 2016]

Name: Solutions

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1. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ conisidered as a metric and topological space in the usual way (as $X \subset \mathbb{R}$) and let $Y = X \times X$ (product topology or topology from the usual $\mathbb{R}^2$ metric). Find the nowhere dense subsets of Y and the subsets of Y that are of first category in Y.

   Solution: X and Y are countable sets. We know that isolated points can't be (in) nowhere dense subsets (or subsets of first category either). [Recall that if $y \in Y$ is isolated then $\{y\}$ is open in Y, and in fact also closed in this case because we have a metric space topology. So the interior of the closure of $\{y\}$ is $\{y\}$ again. If $y \in S \subseteq Y$, then $y \in \bar{S}^\circ$, so that S fails to be nowhere dense.]

   The isolated points of X are all the points $1/n$ with $n \in \mathbb{N}$. (0 is the only non-isolated point.) The isolated points in Y are all the points $(1/n, 1/m)$ with $n, m \in \mathbb{N}$. So every nowhere dense $S \subseteq Y$ must be contained in $(\{0\} \times X) \cup (X \times \{0\})$. This latter is in fact nowhere dense in Y. It is closed (it's complement is the union of all the isolated points) and has no interior — for example an open set in Y that contains $(0, x)$ with $x \in X$ must contain an open set of the form $U \times V$ with $U, V \subset X$ both open, $0 \in U$, $x \in V$. But then it contains $(1/n, x)$ for $n \in \mathbb{N}$ large enough, which is not in $(\{0\} \times X) \cup (X \times \{0\})$ unless $x = 0$. If $x = 0$ we can make n large enough that $1/n \in V$ also and then $(1/n, 1/n) \notin (\{0\} \times X) \cup (X \times \{0\})$ (but $(1/n, 1/n) \in U \times V$).

   A similar argument deals with an open set in Y that contains $(x, 0)$.

   So $(\{0\} \times X) \cup (X \times \{0\})$ and all its subsets are the nowhere dense subsets of Y, and these are also all of first category in Y.

2. Recall that $\ell^1 \subseteq \ell^2$ and so on the space $E = \ell^1$ we have two norms, the usual one $\|\cdot\|_1$ and $\|\cdot\|_2$ (the restriction of the usual norm on ℓ^2).

   (a) Show that the identity operator : $(E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$ is not bounded. [Hint: consider $x^{(n)} = (1, 1, \ldots, 1, 0, 0, \ldots)$ with n ones.]

   Solution: If it was bounded, there would be a constant $M > 0$ such that

   $$\|\mathrm{id}\|_1 = \|x\|_1 \leq M\|x\|_2$$

   for all $x \in \ell^2$. But for $x^{(n)}$ as in the hint,

   $$\|x^{(n)}\|_1 = n \qquad \|x^{(n)}\|_2 = \sqrt{1^2 + 1^2 + \cdots + 1^2} = \sqrt{n}$$

   so that we would have to have

   $$\frac{n}{\sqrt{n}} = \sqrt{n} \leq M$$

   for all n — impossible.

(b) Explain why the open mapping theorem cannot be applied to the (bijective? and bounded?) identity operator : $(E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$.

Solution: It is linear because the vector space operations are the same ones in E whether we take $\|\cdot\|_1$ or $\|\cdot\|_2$. It is bounded because we proved it (if $\|x\|_1 \leq 1$, then $|x_n| \leq 1$ for each n, so that $|x_n|^2 \leq |x_n|$ and so $\sum_n |x_n|^2 \leq \sum_n |x_n| = \|x\|_1 \leq 1$; hence $\|x\|_2 \leq 1$).

The theorem requires the domain and the raget to be both Banach spaces, $E = \ell^1$ is complete in it usual norm $\|\cdot\|_1$. But in the other norm $\|\cdot\|_2$ it cannot be complete — if it was the open mapping theorem would make the inverse map bounded and in (a) we saw that that fails.

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