MA3421 (Functional Analysis 1) Tutorial sheet 8
[December 1, 2016]

Name: Solutions

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1. For measurable functions $f, g\colon X \to \mathbb{K}$ on a measure space (X, Σ, μ), let $[f]$ denote the μ-a.e. equivalance class of f, by which we mean

$$[f] = \{h : h\colon X \to \mathbb{K} \text{ is } \Sigma\text{-measurable and } \mu(\{x \in X : f(x) \neq h(x)\}) = 0\}$$

(and $[g]$ the equivalence class of g).

Show that it make sense to define addition, multiplication by scalars $\lambda \in \mathbb{K}$ and multiplication of equivalence classes by these rules:

$$[f] + [g] = [f + g], \quad \lambda[f] = [\lambda f] \text{ and } [f][g] = [fg].$$

[Hint: You should take it as known that $f + g$, λf and fg are measurable as long as f and g are. Also the union of two sets of measure zero will have measure 0.]

Solution: Suppose $[f] = [f_1]$ and $[g] = [g_1]$, where all of f, f_1, g and g_1 are measurable functions on X. So $E_f = \{x \in X : f(x) \neq f_1(x)\}$ has $\mu(E_f) = 0$ and $E_g = \{x \in X : g(x) \neq g_1(x)\}$ has $\mu(E_g) = 0$.

(The idea of the notation is that E is for exceptional, and the exceptions are small in the sense of measure zero.)

By subadditiviy of μ (or just the hint),

$$\mu(E_f \cup E_g) \leq \mu(E_f) + \mu(E_g) = 0$$

and so $\mu(E_f \cup E_g) = 0$.

Since

$$E_{f+g} = \{x \in X : f(x) + g(x) \neq f_1(x) + g_1(x)\} \subseteq E_f \cup E_g$$

we have $\mu(E_{f+g}) = 0$ or $f + g = f_1 + g_1$ μ-a.e. That means $[f + g] = [f_1 + g_1]$ and so it is not ambiguous to define $[f] + [g] = [f + g]$.

For fg we have basically the same argument because

$$E_{fg} = \{x \in X : f(x)g(x) \neq f_1(x)g_1(x)\} \subseteq E_f \cup E_g.$$

So $\mu(E_{fg}) = 0$ and $[fg] = [f_1 g_1]$.

For $\lambda[f]$ (with $\lambda \in \mathbb{K}$) it is even easier since then

$$E_{\lambda f} = \{x \in X : \lambda f(x) \neq \lambda f_1(x)\} = E_f$$

when $\lambda \neq 0$ (and $E_{\lambda f} = \emptyset$ when $\lambda = 0$). So $[\lambda f] = [\lambda f_1]$.

2. For a fixed $g \in L^t(X, \Sigma, \mu)$ $(1 < t \leq \infty)$ (where again (X, Σ, μ) is a measure space, and really it is $[g]$ that is in $L^t(X, \Sigma, \mu)$) show that for $1 \leq r, s < \infty$ it is possible to define a bounded linear operator

$$M_g \colon L^r(X, \Sigma, \mu) \to L^s(X, \Sigma, \mu)$$

provided $p = r/s \geq 1$ and $sq = t$ (where $1 < q \leq \infty$ is related to p by $1/p + 1/q = 1$) by using the rule

$$M_g([f]) = [gf].$$

[Hint: Use Hölder's inequality. (Aside M_g is called a *multiplication operator*.)]

Solution: The aim is to show first that $gf \in L^s(X, \Sigma, \mu)$, which means to show

$$\int_X |gf|^s \, d\mu < \infty$$

(whenever $f \in L^r(X, \Sigma, \mu)$ and g is as specified).

Using Hölder, we can say

$$\begin{aligned}
\int_X |gf|^s \, d\mu &= \int_X |f|^s |g|^s \, d\mu \\
&\leq \| |f|^s \|_p \| |g|^s \|_q \\
&= \left(\int_X |f|^{sp} \, d\mu \right)^{1/p} \left(\int_X |g|^{sq} \, d\mu \right)^{1/q} \\
&= \left(\int_X |f|^r \, d\mu \right)^{1/p} \left(\int_X |g|^t \, d\mu \right)^{1/q} \\
&= \|f\|_r^{r/p} \|g\|_t^{t/q} = \|f\|_r^s \|g\|_t^s
\end{aligned}$$

As $\|f\|_r < \infty$ and $\|g\|_t < \infty$ we are sure that $gf \in L^s(X, \Sigma, \mu)$ always. But we can also rewrite the inequality we got as $\|gf\|_s^s \leq \|f\|_r^s \|g\|_t^s$ and so we get $\|gf\|_s \leq \|f\|_r \|g\|_t$ (by taking the $1/s$ power).

We can verify that M_g is linear because

$$M_g(f + \lambda h) = g(f + \lambda h) = fg + \lambda gh = M_g(f) + \lambda M_g(h)$$

(for $f, h \in L^r(X, \Sigma, \mu)$, $\lambda \in \mathbb{K}$).

[To be more precise we should use $[f]$ rather than f, and the same for $[gf]$ etc, but the point of Q1 is that we don't really need to worry. Also, why is $g(f + \lambda h) = fg + \lambda gh$? The answer is that the left side is the function with value at $x \in X$ which is $g(x)(f(x) + \lambda h(x))$. Since the values $g(x)$, $f(x)$ and $h(x)$ are just numbers, this is certainly equal to $g(x)f(x) + \lambda g(x)h(x)$, which is the value at x of $fg + \lambda gh$.]

If we rewrite the last form of the inequality as

$$\|M_g(f)\|_s \leq \|g\|_t \|f\|_r$$

(now that M_g is linear) it says that M_g is a bounded operator with operator norm $\|M_g\|_{\text{op}} \leq \|g\|_t$.

3. With the same notation and assumptions as the previous question, assume also that $\mu(X) < \infty$. Then show that if $1 \leq u \leq s$, we can define

$$M_g \colon L^r(X, \Sigma, \mu) \to L^u(X, \Sigma, \mu)$$

by the same rule and again get a bounded linear operator. [Hint: there is a result in the notes that helps a lot.]

Solution: If $\mu(X) < \infty$, then we have $L^s(X, \Sigma, \mu) \subseteq L^u(X, \Sigma, \mu)$ with a bounded (or continuous) inclusion map. (See Examples 2.6.6 (iii).) Combining that fact with Q2, we can get this result immediately. We have a composition of continuous linear maps

$$L^r(X, \Sigma, \mu) \stackrel{M_g}{\to} L^s(X, \Sigma, \mu) \stackrel{\text{inclusion map}}{\hookrightarrow} L^u(X, \Sigma, \mu)$$

(which then will be continuous and linear — hence a bounded linear operator). The end result is a map that sends $f \in L^s(X, \Sigma, \mu)$ to $gf \in L^u(X, \Sigma, \mu)$.

[So we use the same notation M_g for it.]

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