MA3421 (Functional Analysis 1) Tutorial sheet 7
[November 24, 2016]

Name: Solutions

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1. For $1 \leq p < \infty$, show that the set of finitely nonzero sequences
$$c_{00} = \{x = (x_n)_{n=1}^\infty : x_n \in \mathbb{K} \forall n \text{ and } \exists n_0 \in \mathbb{N} \text{ such that } x_n = 0 \forall n > n_0\}$$
is dense in ℓ^p.

   Solution: It is first of all clear that $c_{00} \subset \ell^p$ because for $x = (x_n)_{n=1}^\infty$ with $x_n = 0 \forall n > n_0$, we have
$$\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^{n_0} |x_n|^p + 0 < \infty.$$
Suppose $y = (y_n)_{n=1}^\infty \in \ell^p$, which means that $\|y\|_p^p = \sum_{n=1}^\infty |y_n|^p < \infty$.
If $\delta > 0$, then we can find N so that
$$\sum_{n=n_0+1}^\infty |y_n|^p = \|y\|_p^p - \sum_{n=1}^{n_0} |y_n|^p < \delta^p.$$
If we take $x = (y_1, \ldots, y_{n_0}, 0, 0, \ldots)$, then we have $x \in c_{00}$ and
$$y - x = (0, \ldots, 0, y_{n_0+1}, y_{n_0+2}, \ldots)$$
so that
$$\|y - x\|_p^p = \sum_{n=n_0+1}^\infty |y_n|^p < \delta^p,$$
or $\|y - x\|_p < \delta$.

   So every open ball $B(y, \delta)$ in ℓ^p contains a point of c_{00} and that shows that c_{00} is dense in ℓ^p.

2. Show that ℓ^p is separable for $1 \leq p < \infty$.

   Solution: We need to identify a countable dense subset pf ℓ^p. The subset (subspace) c_{00} won't do because it is certainly uncountable but we can replace it by a rational sequences version (finitely non-zero sequences of rationals).

   We need to explain that a little differently depending on whether we are dealing with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If $\mathbb{K} = \mathbb{R}$ we take those $x \in c_{00}$ such that each $x_n \in \mathbb{Q}$. That is
$$c_{00}^{\mathbb{Q}} = \{q = (q_n)_{n=1}^\infty : q_n \in \mathbb{Q} \forall n \text{ and } \exists n_0 \in \mathbb{N} \text{ such that } q_n = 0 \forall n > n_0\}$$
This is countable because it is a countable union of copies of $\mathbb{Q}^{n_0}$, and these are countable.

   It is dense because the closure (of $c_{00}^{\mathbb{Q}}$ in ℓ^p) includes c_{00}, and therefore the closure of c_{00}, which we know is all of ℓ^p from Q1. To explain in more detail why the closure of $c_{00}^{\mathbb{Q}}$ contains c_{00} we could refer to the fact that $\mathbb{Q}^{n_0}$ is dense in $\mathbb{R}^{n_0}$, or we could just repeat the argument for that:

If $x \in c_{00}$ has $x_n = 0$ for all $n > n_0$, and if $\delta > 0$ is fixed, we can choose $q_1, \ldots, q_{n_0} \in \mathbb{Q}$ with $|q_n - x_n|^p < \delta^p/n_0$ for $n = 1, 2, \ldots, n_0$. Then $q = (q_1, \ldots, q_{n_0}, 0, 0, \ldots) \in c_{00}^{\mathbb{Q}}$ and $\|q - x\|_p^p = \sum_{n=1}^{n_0} |q_n - x_n|^p < \delta^p$, so that $\|q - x\|_p < \delta$.

For the case $\mathbb{K} = \mathbb{C}$, we just need to adjust the rationals q_n to be in $\mathbb{Q} + i\mathbb{Q}$ (complex numbers with rationals as real and imaginary parts). The ideas are no different. We might call the countable set $c_{00}^{\mathbb{Q}+i\mathbb{Q}}$ and it is a union of $\mathbb{Q}^{2n_0}$ for $n_0 = 1, 2, \ldots$ (still countable).

3. Let $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (where $1 \leq p, q \leq \infty$) and $b = (b_n)_{n=1}^\infty \in \ell^q$. Show that it makes sense to define $T_b \colon \ell^p \to \mathbb{K}$ by the formula

$$T_b(x) = T_b\left((x_n)_{n=1}^\infty\right) = \sum_{n=1}^\infty x_n b_n$$

[Hint: Hölder's inequality.]

Solution: Hölder's inequality says that for any $x = (x_n)_{n=1}^\infty \in \ell^p$ and $b = (b_n)_{n=1}^\infty \in \ell^q$, we have

$$\left|\sum_{n=1}^\infty x_n b_n\right| \leq \sum_{n=1}^\infty |x_n b_n| \leq \|x_p\| \|b\| q$$

so that the series we use to define $T_b(x)$ is always convergent (even absolutely convergent) in $\mathbb{K}$.

So the definition makes sense.

4. With the same notation as in the previous question, show that $T_b \colon \ell^p \to \mathbb{K}$ is linear and bounded.

Solution: To show T_b is linear (with $b \in \ell^q$ fixed), take $x, y \in \ell^p$ and $\lambda \in \mathbb{K}$. We aim to check $T_b(x + \lambda y) = T_b(x) + \lambda T_b(y)$. Write out $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$ (and b as in Q3) so that then

$$x + \lambda y = (x_n + \lambda y_n)_{n=1}^\infty$$

$$T_b(x + \lambda y) = \sum_{n=1}^\infty (x_n + \lambda y_n) b_n \sum_{n=1}^\infty x_n b_n + \lambda \sum_{n=1}^\infty y_n b_n = T_b(x) + \lambda T_b(y)$$

(by standard facts about sums and multiples of convergent series).

Once we know that T_b is linear, the inequality in Q3, rewritten as

$$|T_b(x)| \leq \|b\|_q \|x\|_p$$

says that T_b is bounded (with operator norm at most $\|b\|_q < \infty$) as an operator from $(\ell^p . \| \cdot \|_p)$ to $\mathbb{K}$ (with the usual absolute value norm).

Aside. It is in fact true that $\|T_b\|_{\text{op}} = \|b\|_q$. We have shown $\|T_b\|_{\text{op}} \leq \|b\|_q$. If $p = 1$ and $q = \infty$, the argument for the reverse is simpler, but different. In that case we can take x to have 1 in position n and zeroes elsewhere. Maybe it would be simpler to write $x = e_n$ for that x (because it is like a standard basis vector for $\mathbb{K}^N$, but has infinitely many coordinates, or terms). We get $\|e_n\|_1 = 1$ and $|T_b(e_n)| = |b_n|$ so that $|b_n| \leq \|T_b\|_{\text{op}}\|e_n\|_1 = \|T_b\|_{\text{op}}$. As that is true for each n, we get

$$\sup_{n\geq 1} |b_n| \leq \|T_b\|_{\text{op}}$$

or $\|b\|_\infty = \|b\|_q \leq \|T_b\|_{\text{op}}$ (for the case $p = 1$).

For other $p < \infty$ we have to arrange x_n so that $x_n b_n = |b_n|^q$ for each n. If $b_n = 0$ we can just put $x_n = 0$ also and if $b_n \neq 0$ we need

$$x_n = \frac{\bar{b}_n}{|b_n|}|b_n|^{q-1}$$

Then $x = (x_n)_{n=1}^\infty \in \ell^p$ works out because

$$\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty |b_n|^{(q-1)p} = \sum_{n=1}^\infty |b_n|^q = \|b\|_q^q < \infty$$

(using $pq - p = q$ from $1/p + 1/q = 1$). Now $T_b(x) = \sum_{n=1}^\infty x_n b_n = \sum_{n=1}^\infty |b_n|^q = \|b\|_q^q$. So we must have

$$\|b\|_q^q = |T_b(x)| \leq \|T_b\|_{\text{op}}\|x\|_p = \|T_b\|_{\text{op}}\|b\|_q^{q/p}.$$

As long as $b \neq 0$, we can divide across by $\|b\|_q^{q/p}$ to get

$$\|b\|_q^{q-q/p} \leq \|T_b\|_{\text{op}}$$

and that is what we need since $q - q/p = 1$. (If $b = 0$, then $\|T_b\|_{\text{op}} \leq \|b\|_q = 0$ is enough to get $\|T_b\|_{\text{op}} = 0$.)

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