MA3421 (Functional Analysis 1) Tutorial sheet 6
[November 17, 2016]

Name: Solutions

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1. Let $c = \{(x_n)_{n=1}^\infty \in \ell^\infty : \lim_{n\to\infty} x_n \text{ exists}\}$ (the convergent sequences of scalars). Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. (We consider $X \subseteq \mathbb{R}$ with its usual topology, which makes it compact.) Show that it is possible to define a linear surjective isometry $T \colon c \to C(X)$ by setting $T((x_n)_{n=1}^\infty) = f$ where
$$f(1/n) = x_n$$
and $f(0) = \lim_{n\to\infty} x_n$.

Solution: To show that T is well-defined, we need to know that $f \colon X \to \mathbb{K}$ is continuous (no matter which $(x_n)_{n=1}^\infty \in c$ is considered). But the points $1/n$ are isolated in X and every $f \colon X \to \mathbb{K}$ is continuous at each $1/n$. The only issue is continuity at 0, which requitres $\lim_{n\to\infty} f(1/n) = f(0)$ (or that for $\varepsilon > 0$ given we have $|f(1/n) - f(0)| < \varepsilon$ for all n large). Since we have $f(0) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(1/n)$, this is true.

Next T is linear because, if $x = (x_n)_{n=1}^\infty \in c$ and $y = (y_n)_{n=1}^\infty \in c$
$$T(x+y) = T\left((x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty\right) = T\left((x_n + y_n)_{n=1}^\infty\right)$$
has
$$T(x+y)\left(\frac{1}{n}\right) = x_n + y_n = T(x)\left(\frac{1}{n}\right) + T(y)\left(\frac{1}{n}\right).$$
Also,
$$T(x+y)(0) = \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + + \lim_{n\to\infty} y_n = T(x)(0) + T(y)(0).$$

So $T(x+y)$ is the same function as $T(x) + T(y)$.

For $\lambda \in \mathbb{K}$ (and still $x = (x_n)_{n=1}^\infty \in c$) we have
$$T(\lambda x) = T\left((\lambda x_n)_{n=1}^\infty\right)$$
so that
$$T(\lambda x)\left(\frac{1}{n}\right) = \lambda x_n = \lambda T(x)\left(\frac{1}{n}\right)$$
Taking limits, we get
$$T(\lambda x)(0) = \lim_{n\to\infty} \lambda x_n = \lambda \lim_{n\to\infty} x_n = \lambda T(x)(0).$$

To $T(\lambda x)$ is the same function on X as $\lambda T(x)$. That is $T(\lambda x)` = \lambda T(x)$.

Starung with $f \in C(X)$ we can get a convergent sequecnce $x = (x_n)_{n=1}^\infty$ with $T(x) = f$ by taking $x_n = f(1/n)$. So $T \colon c \to C(X)$ is surjective.

Finally we show that $\|T(x)\| = \|x\|$ for each $x \in c$.

The norm $\|x\|$ we use on c is the the one we get by restricting the norm from ℓ^∞, that is

$$\|x\|_\infty = \sup_n |x_n|.$$

The norm of $f = T(x) \in C(X)$ is also a supremum

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Since $f(0) = \lim_{n\to\infty} f(1/n)$ holds for $f \in C(X)$, we have

$$\|f\|_\infty = \sup_{x \in X} |f(x)| = \sup_{0 \neq x \in X} |f(x)| = \sup_{n \in \mathbb{N}} |f(1/n)| = \sup_n |x_n|$$

when $f = T(x) - T\left((x_n)_{n=1}^\infty\right)$. So T is an isometry.

Richard M. Timoney