MA3421 (Functional Analysis 1) Tutorial sheet 5

[October 27, 2016]

Name: Solutions

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1. Let $x_n \in \ell^2$ denote the sequence

$$x_n = (0, \ldots, 0, \frac{1}{n}, 0, 0, \ldots)$$

where the nth term is $1/n$ and all other terms are 0. More formally $x_n = (x_{n,j})_{j=1}^\infty$ where

$$x_{n,j} = \begin{cases} \frac{1}{n} & \text{if } j = n \\ 0 & \text{for } j \neq n \end{cases}$$

Write out the partial sums $s_2 = x_1 + x_2$ and $s_3 = x_1 + x_2 + x_3$ of the series $\sum_{n=1}^\infty x_n$ in ℓ^2.

Solution:

$$\begin{aligned} s_2 &= x_1 + x_2 \\ &= (1, 0, 0, \ldots) + \\ &\quad (0, \frac{1}{2}, 0, \ldots) \\ &= (1, \frac{1}{2}, 0, 0, \ldots) \end{aligned}$$

$$\begin{aligned} s_3 &= x_1 + x_2 + x_3 \\ &= s_2 + x_3 \\ &= (1, \frac{1}{2}, 0, \ldots) + \\ &\quad (0, 0, \frac{1}{3}, 0, \ldots) \\ &= (1, \frac{1}{2}, \frac{1}{3}, 0, \ldots) \end{aligned}$$

Then write out the general partial sum $s_n = \sum_{k=1}^n x_k$.

Solution:

$$s_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)$$

2. Show that the series $\sum_{n=1}^{\infty} x_n$ (from the previous question) converges in ℓ^2 and find it's sum explicitly. However, show that the series does **not** converge absolutely.

   Solution: Consider $s = (1, \frac{1}{2}, \frac{1}{3}, \ldots) = \left(\frac{1}{n}\right)_{n=1}^{\infty}$ (as the likely sum).

   First $s \in \ell^2$ because
$$\sum_{n=1}^{\infty} \left|\frac{1}{n}\right|^2 = \sum_{n=1}^{\infty} \frac{1}{n}^2 < \infty$$

   Next we can see that $\lim_{n\to\infty} s_n = s$ in the ℓ^2-norm because
$$s_n - s = (0, 0, \ldots, 0, -\frac{1}{n+1}, -\frac{1}{n+2}, \ldots)$$
   and so
$$\|s_n - s\|_2 = \left(\sum_{j=n+1}^{\infty} \frac{1}{j^2}\right)^{1/2} \to 0$$
   as $n \to \infty$ (because $\sum_{n=1}^{\infty} \frac{1}{n}^2 < \infty$ and the expression inside the square root is the difference $\sum_{n=1}^{\infty} \frac{1}{n}^2 = \sum_{1}^{n} \frac{1}{j^2}$ between the infinite sum and the n^{th} partial sum.)

   Finally, absolute convergence would mean $\sum_{n=1}^{\infty} \|x_n\|_2 < \infty$ and in this case
$$\|x_n\|_2 = (0^2 + \cdots 0^2 + \frac{1}{n^2} + 0^2 + \cdots)^{1/2} = \frac{1}{n}$$
   so that
$$\sum_{n=1}^{\infty} \|x_n\|_2 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$
   (harmonic series diverges). So it is not absolutely convergent.

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