MA3421 (Functional Analysis 1) Tutorial sheet 4
[October 27, 2016]

Name: Solutions

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1. Let $X = \{1, 2, 3\}$ and let $\mathcal{S} = \{\{1\}, \{1, 2\}\}$. Find the topology on X generated by (the sub-base) $\mathcal{S}$.

   Solution: To get a base from (the sub-base) $\mathcal{S}$, we can consider all finite intersections of sets from $\mathcal{S}$, including the empty intersection which we take to be X. But that gives us

   $$\mathcal{B} = \{\{1\}, \{1, 2\}, X\}.$$

   Then the topology for that base is all unions of collections from $\mathcal{B}$, including the empty union, which means the empty set. We get the topology

   $$\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}.$$

   We can see easily that this is indeed a topology (though we know that it must be because of proofs we have previously done). Certainly $\emptyset, X \in \mathcal{T}$ and (arbitrary) and (finite) intersections of sets in $\mathcal{T}$ are again in $\mathcal{T}$.

2. Let $X = \{1, 2, 3, 4\}$ and let $f_1, f_2 \colon X \to \mathbb{R}$ be the characteristic functions $f_1 = \chi_{\{1\}}$, $f_2 = \chi_{\{1,2\}}$. Find the weak topology on X generated by f_1 and f_2.

   Solution: We know that a characteristic function of a subset of a topological space will be continuous if and only if the subset is clopen (both open and closed). [If you consider the inverse images of the open intervals $(-1/2, 1/2)$ and $(1/2, 3/2)$ under (say) f_1, you will get $X \setminus \{1\}$ and $\{1\}$ — both of which need to be open if f_1 is going to be continuous.]

   So we need these sets to be open

   $$\{\{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}\}$$

   Taking (finite) intersections and (arbitrary) unions of these we get to

   $$\{\emptyset, \{1\}, \{2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{2\}, \{1, 3, 4\}, X\}$$

   and that is the topology.

3. Let X be a first countable topological space. Show that for each $x_0 \in X$ there is a neighbourhood base $\mathscr{B}_{x_0} = \{B_1', B_2', \ldots\}$ at x_0 such that $B_1' \supseteq B_2' \supseteq B_3' \supseteq \cdots$. [Hint: Look at the proof that characterizes closures of subsets of first countable spaces via sequences.]

   Solution: Fix a point $x_0 \in X$. Since X is first countable, there is a countable neighbourhood base $\mathcal{B}_{x_0}$ at x_0. We can write the sets in $\mathcal{B}_{x_0}$ in a list (finite or infinite) $\mathcal{B}_{x_0} = \{B_1, B_2, \ldots\}$.

The list can't be empty as X is a neighbourhood of x_0 and so there must be $B \in \mathcal{B}_{x_0}$ with $x \in B \subseteq X$. If the list is finite, we can treat it as infinite by repeating the last set infinitely often. So, if $\mathcal{B}_{x_0} = \{B_1, B_2, \ldots, B_n\}$ let $B_j = B_n$ for $j = n+1, n+2, \ldots$.

Now define $B_1' = B_1$, $B_2' = B_1 \cap B_2$ and in general

$$B_n' = \bigcap_{i=1}^{n} B_i = B_1 \cap B_2 \cap \cdots \cap B_n$$

for $n \geq 1$. Then we can see that each B_n' is a neighbourhood of x_0 because the intersection of two neighbourhoods (of x_0) is again a neighbourhood. (So that extends to finite intersections by induction, or we can use that $B_n' = B_{n-1}' \cap B_n$ for $n > 1$.)

Let $\mathcal{B}_{x_0}' = \{B_n' : n \in \mathbb{N}\}$. We claim it is a neighbourhood base at x_0. Indeed we have already noted that each B_n' is a neighbourhood of x_0. If N is any neighbourhood of x_0, then we know that there is n so that $B_n \subseteq N$ (because $\mathcal{B}_{x_0}$ is a neighbourhood base). It follows that $B_n' \subseteq B_n \subseteq N$.

Clearly $B_1' \supseteq B_2' \supseteq B_3' \supseteq \cdots$.

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