MA3421 (Functional Analysis 1) Tutorial sheet 3
[October 20, 2016]

Name: Solutions

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1. Let (X, d) be a metric space. Define $\rho\colon X \times X \to [0, \infty)$ by

$$\rho(x, y) = \min(d(x, y), 1)$$

Show that ρ is also a metric on X and that the metric topology for d is the same as the metric topology for ρ.

Solution: We need to check that ρ satisfies the conditions to be a metric (and we know that d does).

(i) $\rho(x, y) \geq 0$ (all $x, y \in X$) and $\rho(x, y) = 0 \Rightarrow x = y$.
Since $d(x, y) \geq 0$, we have $\rho(x, y) \geq 0$. If $\rho(x, y) = 0$, then $d(x, y) = 0$ and so $x = y$.

(ii) $\rho(y, x) = \rho(x, y)$ (for all $x, y \in X$)
True since $d(y, x) = d(x, y)$.

(iii) Triangle inequality: $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.
Consider two cases: First case $d(x, y) < 1$ and $d(y, z) < 1$. Then

$$\rho(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \rho(x, y) + \rho(y, z).$$

Second case is where one or both of $d(x, y)$, $d(y, z)$ is at least 1. Then one or both of $\rho(x, y)$, $\rho(y, z)$ is 1 and so

$$\rho(x, y) + \rho(y, z) \geq 1 \geq \rho(x, z).$$

For the topology part, if $r < 1$ then

$$B_d(x, r) = \{y \in X : d(x, y) < r\} = B_\rho(x, r) = \{y \in X : \rho(x, y) < r\}$$

If $U \subset X$ is open for the d-topology then for each $x \in U$ there is some $r > 0$ with $B_d(x, r) \subset U$. But we decrease r and make $r < 1$ and this is still true. Then $B_\rho(x, r) = B_d(x, r) \subset U$. That shows U is open in the ρ-topology.

The reverse, showing that open sets in the ρ-topology are open in the d-topology, is just the same.

2. Let (X_n, d_n) be metric spaces (for $n = 1, 2, \ldots$) and let $X = \prod_{n=1}^{\infty} X_n$ be the (infinite) cartesian product set. Let $\rho_n \colon X_n \times X_n \to [0, \infty)$ be defined by $\rho_n(x_n, y_n) = \min(d_n(x_n, y_n), 1)$ for $x_n, y_n \in X_n$.

   Show that it is possible to define a metric ρ on the set X by

$$\rho\left((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n)$$

   Solution: We need to check that ρ satisfies the conditions to be a metric (and that the series always converges).

   (i) Since $\rho_n(x_n, y_n) \leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

   (So the formula defining ρ makes sense, as the sum of a series of positive terms dominated by a convergent series.)

   Clearly $\rho(x, y) \geq 0$ (for $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$) since $\rho_n(x_n, y_n) \geq 0$. If $\rho(x, y) = 0$, we must have $\rho_n(x_n, y_n) = 0$ for all n and so $x_n = y_n$ because ρ_n is a metric (see Q1). Hence

$$x = (x_n)_{n=1}^{\infty} = (y_n)_{n=1}^{\infty} = y$$

   (if $\rho(x, y) = 0$).

   (ii) $\rho(x, y) = \rho(y, x)$ always because $\rho_n(x_n, y_n) = \rho_n(y_n, x_n$ (ρ_n is a metric).

   (iii) (Triangle inequality)

   If we have $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ (all in X), then

$$\begin{aligned}
\rho(x, z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, z_n) &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\rho_n(x_n, y_n) + \rho_n(y_n, z_n)\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n) + \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(y_n, z_n) \\
&= \rho(x, y) + \rho(y, z)
\end{aligned}$$

   (using the triangle inequality for each ρ_n).

3. For the same X and ρ as in question 2, let $\pi_n \colon X \to X_n$ be the coordinate projection

$$\pi_n\left((x_m)_{m=1}^{\infty}\right) = x_n.$$

   Show that π_n is continuous from (X, ρ) to (X_n, d_n) for each $n = 1, 2, \ldots$.

Solution: Fix n and we use the ε-δ definition of continuity. Let $\varepsilon > 0$ be given.

Put $\delta = \min(\varepsilon, 1)/2^n$. Then

$$\rho\left((x_m)_{m=1}^\infty, (y_m)_{m=1}^\infty\right) < \delta$$

implies

$$\frac{1}{2^n}\rho_n(x_n, y_n) < \frac{\min(\varepsilon, 1)}{2^n} \Rightarrow \rho_n(x_n, y_n) < \min(\varepsilon, 1) \Rightarrow d_n(x_n, y_n) < \varepsilon.$$

That means that π_n is (uniformly) continuous, in particular continuous at each $x = (x_m)_{m=1}^\infty \in X$.

There are other proofs.

Since continuity means that $\pi_n^{-1}(U)$ is open in (X, ρ) for each $U \subseteq X_n$ open, and the open sets in X_n are the same whether we use the metric d_n or the metric ρ_n, we can use ρ_n on X_n.

Now

$$\rho(x, y) = \rho\left((x_m)_{m=1}^\infty, (y_m)_{m=1}^\infty\right) = \sum_{m=1}^\infty \frac{1}{2^m}\rho_m(x_m, y_m) \geq \frac{1}{2^n}\rho_n(x_n, y_n)$$

we have

$$\rho_n(\pi_n(x), \pi_n(y)) = \rho_n(x_n, y_n) \leq 2^n \rho(x, y).$$

This means that $\pi_n \colon (X, \rho) \to (X_n, \rho_n)$ is what is called a Lipschitz map — satisfies

$$\rho_n(\pi_n(x), \pi_n(y)) \leq C\rho(x, y)$$

for some $C \geq 0$.

Given $\varepsilon > 0$ we can take $\delta = \varepsilon/(C + 1)$ [just adding the 1 to make sure no division by 0] and we have

$$\rho(x, y) < \delta \Rightarrow \rho_n(\pi_n(x), \pi_n(y)) \leq C\rho(x, y) \leq \frac{C}{C+1}\varepsilon < \varepsilon.$$

That means we have uniform continuity again (with ρ and ρ_n this time, rather that ρ and d_n as before).

It is in fact true that, in the last question, the metric topology for (X, ρ) is the same as the product topology. Since each π_n is continuous on (X, ρ), the product topology is weaker than the ρ-topology (as the product topology is defined to be the weakest, or smallest, topology making all π_n continuous.

If $U \subseteq X$ is open in the product topology and $x = (x_m)_{m=1}^\infty \in U$. then we know that there are finitely many open sets

$$V_j \subseteq X_{n_j} \quad (j = 1, 2, \ldots, k)$$

such that

$$x \in \bigcap_{j=1}^{k} \pi_{n_j}^{-1}(V_j) \subseteq U.$$

We can take the indices $n_1, n_2, \ldots, n_k$ to be all different. If we then put $W_{n_j} = V_j$ for $j = 1, 2, \ldots, k$ and $W_n = X_n$ for all other n we have

$$x \in \prod_{n=1}^{\infty} W_n = \bigcap_{j=1}^{k} \pi_{n_j}^{-1}(V_j) \subseteq U.$$

If $N = \max_{1 \leq j \leq k} n_j$, then we have $W_n \subseteq X_n$ open for $n = 1, 2, \ldots, N$, and $W_n = X_n$ for $n > N$.

Say $x = (x_n)_{n=1}^{\infty}$. Then $x_n \in W_n$ for all n.

So there is $\delta_n > 0$ for $n = 1, 2, \ldots, N$ so that

$$B_{\rho_n}^{X_n} = \{y \in X_n : \rho_n(x_n y) < \delta_n\} \subseteq W_n$$

Put

$$\delta = \frac{1}{2^N} \min(\delta_1, \delta_2, \ldots, \delta_N).$$

Then $B_\rho^X(x, \delta) \subset \prod_{n=1}^{\infty} W_n \subseteq U$ because if $y = (y_n)_{n=1}^{\infty} \in B_\rho^X(x, \delta)$,

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n) < \delta$$

and that implies for $n = 1, 2, \ldots, N$

$$\rho_n(x_n, y_n) < 2^N \delta = \min(\delta_1, \delta_2, \ldots, \delta_N) \Rightarrow \rho_n(x_n, y_n) < \delta_n \Rightarrow y_n \in B_{\rho_n}^{X_n} \subseteq W_n,$$

hence $(y_1, y_2, \ldots, y_N) \in W_1 \times W_2 \times \cdots \times W_N \Rightarrow y \in \prod_{n=1}^{\infty} W_n$.

So U is open in (X, ρ).

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