MA3421 (Functional Analysis 1) Tutorial sheet 2
[October 13, 2016]

Name: Solutions

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1. Let X and Y be topological spaces (so that $\mathcal{T}_X$ is the topology for X and $\mathcal{T}_Y$ the topology for Y). Let $Z = X \times Y$. Show that
$$\mathcal{B}_Z = \{U \times V : U \subseteq X \text{ open and } V \subseteq Y \text{ open}\}$$
is a base for some topology on Z. (That topology is known as the product topology.)

   Solution: We need to check just two properties for it to be a base (for some topology on Z):

   (a) $\bigcup_{U \times V \in \mathcal{B}_Z} U \times V = Z$
   But $U = X$ and $V = Y$ are allowed and so $Z = X \times Y \in \mathcal{B}_Z$. So Z is contained in the union (and all the sets $U \times V \in \mathcal{B}_Z$ are contained in Z). So the union is exactly all of Z.

   (b) $B_1 \in \mathcal{B}_Z$, $B_2 \in \mathcal{B}_Z$, $z \in B_1 \cap B_2$ implies there is $B_3 \in \mathcal{B}_Z$ with $z \in B_3 \subseteq B_1 \cap B_2$.
   But if (say) $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ then $B_1 \cap B_2 = (U_1 \cap U_2) \times (V_1 \cap V_2)$ and so $B_1 \cap B_2 \in \mathcal{B}_Z$ because $U_1 \cap U_2$ is open in X and $V_1 \cap V_2$ is open in Y. So we can take $B_3 = B_1 \cap B_2$.

2. In $\mathbb{R}^2$ with the usual Euclidean metric d, consider the open ball $B((x, y), r)$ and the (open) square $S((x, y), r) = (x - r, x + r) \times (y - r, y + r)$. Show that
$$B((x, y), r) \subseteq S((x, y), r) \text{ and } S((x, y), r/\sqrt{2}) \subseteq B((x, y), r).$$

   Solution: A picture is probably a help here, but if $(a, b) \in B((x, y), r)$, that means
$$d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2} < r.$$
   Since squares are not negative, for any such (a, b) we have
$$|a - x| = \sqrt{(a - x)^2} \leq \sqrt{(a - x)^2 + (b - y)^2} < r$$
   and so $a \in (x - r, x_r)$. Similarly $b \in (y - r, y + r)$ and so we get
$$(a, b) \in S((x, y), r).$$
   That proves $B((x, y), r) \subseteq S((x, y), r)$.
   For the other inclusion, if $(a, b) \in S((x, y), r/\sqrt{2})$, we have $|a - x| < r/\sqrt{2}$ and $|b - y| < r/\sqrt{2}$ and so
$$d((a, b), (x, y)) = \sqrt{(a - x)^2 + (b - y)^2} < \sqrt{\frac{r^2}{2} + \frac{r^2}{2}} = r.$$
   So $(a, b) \in B((x, y), r)$ and that shows $S((x, y), r/\sqrt{2}) \subseteq B((x, y), r$.

3. For $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, show that the usual topology (arising from the usual Euclidean metric d) coincides with the product topology (where $\mathbb{R}$ has its usual toplogy). [Hint: Question 2 should do most of the work.]

   Solution: We show that if $O \subseteq \mathbb{R}^2$ is open in the usual topology, then it is open in the product topology. (Later we show the opposite, that is O is open in the product topology then it is also open in the usual topology.)

   If O is open in the usual topology and $(x_0, y_0) \in O$, then there is $r > 0$ with $B((x_0, y_0), r) \subseteq O$. But then, using Question 2,

   $$S((x_0, y_0), r/\sqrt{2}) = \left(x_0 - \frac{r}{\sqrt{2}}, x_0 + \frac{r}{\sqrt{2}}\right) \times \left(y_0 - \frac{r}{\sqrt{2}}, y_0 + \frac{r}{\sqrt{2}}\right) \subseteq B((x_0, y_0), r) \subseteq O$$

   and that means that (x_0, y_0) is in the interior of O for the product topology. As that is true of every $(x_0, y_0) \in O$, O must be open for the product topology.

   [Another way to argue would be to say that there is an $r_{(x_0,y_0)}$ for each $(x_0, y_0) \in O$ with

   $$S((x_0, y_0), r_{(x_0,y_0)}/\sqrt{2}) \subseteq O$$

   and then

   $$O = \bigcup_{(x_0,y_0)} \left(x_0 - \frac{r_{(x_0,y_0)}}{\sqrt{2}}, x_0 + \frac{r_{(x_0,y_0)}}{\sqrt{2}}\right) \times \left(y_0 - \frac{r_{(x_0,y_0)}}{\sqrt{2}}, y_0 + \frac{r_{(x_0,y_0)}}{\sqrt{2}}\right)$$

   has to be open in the product topology — but this is perhaps too long-winded and repeating arguments we had already.]

   If O is open in the product topology and $(x_0, y_0) \in O$, then there are $U, V \subseteq \mathbb{R}$ open with

   $$(x_0, y_0) \in U \times V \subseteq O.$$

   But then there is $r_1 > 0$ with $(x_0 - r_1, x_0 + r_1) \subset U$ (because the ball of radius r_1 about x_0 in $\mathbb{R}$ is the interval $(x_0 - r_1, x_0 + r_1)$. And there is $r_2 > 0$ with $(y_0 - r_2, y_0 + r_2) \subset V$. If we take $r = \min(r_1, r_2)$, then we get

   $$S((x_0, y_0), r) = (x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r) \subseteq (x_0 - r_1, x_0 + r_1) \times (y_0 - r_2, y_0 + r_2) \subseteq U \times V \subseteq O.$$

   Then, by Question 2,

   $$B((x_0, y_0), r) \subseteq S((x_0, y_0), r) \subseteq O.$$

   That is what we need for O to be open in the usual metric toplogy (that we can fit inside the set a ball of positive radius about any given point).

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