MA3421 (Functional Analysis 1) Tutorial sheet 1

[October 6, 2016]

Name: Solution

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1. In a metric space (X, d), show that a closed ball $\bar{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ ($x_0 \in X$, $r \geq 0$) must be a closed set. [Explanation: you should prove this using the definition of closed set as the complement of an open set and of an open set in a metric space.]

   Solution: Let $x \in X \setminus \bar{B}(x_0, r)$. So $d(x, x_0) > r$. Take $\delta = d(x, x_0) - r$ (which has $\delta > 0$).

   We claim that $B(x, \delta) \subset X \setminus \bar{B}(x_0, r)$, or that $B(x, \delta) \cap \bar{B}(x_0, r) = \emptyset$. If $y \in B(x, \delta) \cap \bar{B}(x_0, r)$, then that means $d(y, x) < \delta$ and $d(y, x_0) \leq r$. It implies (using the triangle inequality for d and symmetry of d)

   $$d(x, x_0) \leq d(x, y) + d(y, x_0) = d(y, x) + d(y, x_0) < r + \delta = d(x, x_0).$$

   As $d(x, x_0) < d(x, x_0)$ can't be true, there is no such y.

   From $B(x, \delta) \subset X \setminus \bar{B}(x_0, r)$, we see that x is an interior point of $X \setminus \bar{B}(x_0, r)$. As this is true of each $x \in X \setminus \bar{B}(x_0, r)$, $X \setminus \bar{B}(x_0, r)$ must be open and then $\bar{B}(x_0, r)$ has to be closed (by the definition of 'closed').

2. Consider the space $X = \{1/n : n \in \mathbb{N}\}$ with the usual absolute value distance d (from $\mathbb{R}$). Show that the metric topology is the discrete topology (where all subsets are open).

   Solution: Fix $1/n_0 \in X$. If $n_0 = 1$, then all other points in X are $\leq 1/2$. If $n_0 > 1$, the other points of X are all either $\geq 1/(n_0 - 1)$ or $\leq 1/(n_0 + 1)$ and so the nearest other point is always $1/(n_0 + 1)$. If we take

   $$r = \frac{1}{n_0} - \frac{1}{n_0 + 1}$$

   then $B(1/n_0, r) = \{1/n_0\}$ is a singleton set and is open.

   As each singleton subset of X is open, so is every subset — because every $S \subseteq X$ is the union of the singleton sets it contains ($S = \bigcup_{s \in S} \{s\}$) and arbitrary unions of open sets are open.

3. For the same (X, d) as in the previous question, find

   $$B\left(\frac{1}{3}, \frac{1}{6}\right), \bar{B}\left(\frac{1}{3}, \frac{1}{6}\right)$$

   and the closure of $B(1/3, 1/6)$.

Solution: We know

$$d\left(\frac{1}{2},\frac{1}{3}\right) = \left|\frac{1}{2}-\frac{1}{3}\right| = \frac{1}{6}$$

and so $1/2 \notin B\left(\frac{1}{3},\frac{1}{6}\right)$. However $d(1/4, 1/3) = 1/12 < 1/6$, $d(1/5, 1/3) = 2/15 < 1/6$ but $d(1/6, 1/3) = 1/6$.

So

$$B\left(\frac{1}{3},\frac{1}{6}\right) = \left\{\frac{1}{3},\frac{1}{4},\frac{1}{5}\right\}$$

while

$$\bar{B}\left(\frac{1}{3},\frac{1}{6}\right) = \left\{\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{2},\frac{1}{6}\right\}.$$

As every subset of X is open, so also is every subset closed (complement will be open) and so every subset is its own closure (smallest closed subset of X containing any set $S \subseteq X$ will be S).

So the closure of $B\left(\frac{1}{3},\frac{1}{6}\right)$ is itself.

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