XMA34221

Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS & SS Mathematics **Trinity Term 2015**
JS & SS TSM Mathematics

Module MA3421— Functional Analysis I

Day PLACE TIME

Prof. R. Timoney

Instructions to Candidates:

Credit will be given for the best 3 questions answered.

All questions have equal weight.

'Formulae & tables' are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used

In the questions $\mathbb{K}$ denotes one of $\mathbb{R}$ or $\mathbb{C}$.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) [6 points] If $(X, \mathscr{T})$ is a topological space define the concept of a base $\mathcal{B}$ for $\mathscr{T}$. What does it mean for $\mathscr{T}$ to be second countable?

Solution:

Definition 1.1. If $(X, \mathscr{T})$ is a topological space, then a subfamily $\mathcal{B} \subseteq \mathscr{T}$ of the open sets is called a *base (for the open sets)* of the topology if

$$x \in U \subseteq X, U \text{ open} \Rightarrow \exists B \in \mathcal{B} \text{ with } x \in B \subset U$$

Definition 1.2. A topological space $(X, \mathscr{T})$ is called *second countable* if there exists a base $\mathcal{B}$ for the topology where $\mathcal{B}$ is a countable collection of sets.

(b) [7 points] If $\mathscr{T}$ is the discrete topology on X, show that $\mathscr{T}$ is second countable if and only if X is countable.

Solution: If X has the discrete topology, then each singleton subset $\{x\}$ is open and so in any base there must be a basic open set B with $x \in B \subseteq \{x\}$. That means $B = \{x\}$ is in the base. If the base is countable then X has to be countable. Conversely if X is countable and discrete $\mathcal{B} = \{\{x\} : x \in X\}$ is a countable base for X.

(c) [7 points] Define what it means for a toplogical space to be separable. Prove that second countable topological spaces are always separable. Give an example of a separable topological space which is not second countable.

Solution:

Definition 1.3. A topological space $(X, \mathscr{T})$ is called *separable* if there exists a countable subset S of X that is dense in X.

Theorem 1.4. *Second countable topological spaces are always separable.*

Proof. Let $(X, \mathscr{T})$ be a second countable topological space. Let $\mathcal{B}$ be a countable base (for the open sets of) the topology. For each nonempty $B \in \mathcal{B}$, choose an element $x_B \in B$. Let $S = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$. [In fact, if the empty set is included in $\mathcal{B}$, we could remove the empty set from $\mathcal{B}$ and still have a countable base.]

Now S is a countable set (because the map $: \mathcal{B} \setminus \{\emptyset\} \to S : B \mapsto x_B$ is a surjective map from a countable set to S). We claim S is dense. If not $\bar{S} \neq X$ and there is $x \in X \setminus \bar{S}$. Since the complement of a closed set is open, there must be $B \in \mathcal{B}$ with $x \in B \subseteq X \setminus \bar{S}$. Now $B \neq \emptyset$ and so $x_B \in B \Rightarrow x_B \in X \setminus \bar{S}$. But $S \subseteq \bar{S} \Rightarrow X \setminus S \supseteq X \setminus \bar{S}$ and so we conclude $x_B \in X \setminus S$. But this contradicts $x_B \in S$. Hence S is countable dense.

Thus X is second countable. □

Example 1.5. We introduce a *very unusual* topology on the set $\mathbb{R}$ or real numbers by taking as a base all intervals of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ (where $a < b$, $a, b \in \mathbb{R}$).

The set of real numbers with this topology is called the *Sorgenfrey line*.

This is separable because the rationals are dense in this topological space.

It is not second countable because if $\mathcal{B}$ is any base for the open sets in this topology, then for each $x \in \mathbb{R}$ we have $x \in [x, x + 1) =$ and open set in this space. So there must exist $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x + 1)$. Notice then that x has to be the smallest element of B_x, thus guaranteeing that $\{B_x : x \in \mathbb{R}\}$ is an uncountable subset of $\mathcal{B}$. So there is no countable base for the topology.

2. (a) [8 points] Let $(E, \|\cdot\|)$ be a normed space. Show E is a Banach space if and only if each absolutely convergent series $\sum_{n=1}^{\infty} x_n$ of terms $x_n \in E$ is convergent in E.

Solution:

Proof. Assume E is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then the parial sums of this series of positive terms

$$S_n = \sum_{j=1}^{n} \|x_j\|$$

must satisfy the Cauchy criterion. That is for $\varepsilon > 0$ given there is N so that $|S_n - S_m| < \varepsilon$ holds for all $n, m \geq N$. If we take $n > m \geq N$, then

$$|S_n - S_m| = \left| \sum_{j=1}^{n} \|x_j\| - \sum_{j=1}^{m} \|x_j\| \right| = \sum_{j=m+1}^{n} \|x_j\| < \varepsilon.$$

Then if we consider the partial sums $s_n = \sum_{j=1}^{n} x_j$ of the series $\sum_{n=1}^{\infty} x_n$ we see that for $n > m \geq N$ (same N)

$$\|s_n - s_m\| = \left\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \right\| = \left\| \sum_{j=m+1}^{n} x_j \right\| \leq \sum_{j=m+1}^{n} \|x_j\| < \varepsilon.$$

It follows from this that the sequence $(s_n)_{n=1}^{\infty}$ is Cauchy in E. As E is complete, $\lim_{n\to\infty} s_n$ exists in E and so $\sum_{n=1}^{\infty} x_n$ converges.

For the converse, assume that all absolutely convergent series in E are convergent. Let $(u_n)_{n=1}^{\infty}$ be a Cauchy sequence in E. Using the Cauchy condition with $\varepsilon = 1/2$ we can find $n_1 > 0$ so that

$$n, m \geq n_1 \Rightarrow \|u_n - u_m\| < \frac{1}{2}.$$

Next we can (using the Cauchy condition with $\varepsilon = 1/2^2$) find $n_2 > 1$ so that

$$n, m \geq n_2 \Rightarrow \|u_n - u_m\| < \frac{1}{2^2}.$$

We can further assume (by increasing n_2 if necessary) that $n_2 > n_1$. Continuing in this way we can find $n_1 < n_2 < n_3 < \cdots$ so that

$$n, m \geq n_j \Rightarrow \|u_n - u_m\| < \frac{1}{2^j}.$$

onsider now the series $\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{\infty}(u_{n_{j+1}} - u_{n_j})$. It is absolutely convergent because

$$\sum_{j=1}^{\infty} \|x_j\| = \sum_{j=1}^{\infty} \|u_{n_{j+1}} - u_{n_j}\| \le \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

By our assumption, it is convergent. Thus its sequence of partial sums

$$s_J = \sum_{j=1}^{J}(u_{n_{j+1}} - u_{n_j}) = u_{n_{J+1}} - u_{n_1}$$

has a limit in E (as $J \to \infty$). It follows that

$$\lim_{J\to\infty} u_{n_{J+1}} = u_{n_1} + \lim_{J\to\infty}(u_{n_{J+1}} - u_{n_1})$$

exists in E. So the Cauchy sequence $(u_n)_{n=1}^{\infty}$ has a convergent subsequence. By a Lemma E is complete. □

(b) [12 points] Define the sequence space ℓ^p and the norm $\|\cdot\|_p$ on it for $1 \le p < \infty$. Outline the steps required to show for $1 < p < \infty$ that $\|\cdot\|_p$ is indeed a norm and that $(\ell^p, \|\cdot\|_p)$ is a Banach space.

Solution:

Definition 2.1. For $1 \le p < \infty$, ℓ^p denotes the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ which satisfy

$$\sum_{n=1}^{\infty} |a_n|^p < \infty.$$

It is a Banach space in the norm

$$\|(a_n)_n\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$$

To prove it is a norm we rely on Young's inequality:

Lemma 2.2. *Suppose $1 < p < \infty$ and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for } a, b \ge 0.$$

to show

Lemma 2.3 (Hölder's inequality). *Suppose $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$ this is interpreted to mean $q = \infty$ and values of p and q satisfying this relationship are called* **conjugate exponents***). For $(a_n)_n \in \ell^p$ and $(b_n)_n \in \ell^q$,*

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \|(a_n)_n\|_p \, \|(b_n)_n\|_q.$$

(This means both that the series on the left converges and that the inequality is true.)

and

Lemma 2.4 (Minkowski's inequality). *If $x = (x_n)_n$ and $y = (y_n)_n$ are in ℓ^p ($1 \leq p \leq \infty$) then so is $(x_n + y_n)_n$ and*

$$\|(x_n + y_n)_n\|_p \leq \|(x_n)_n\|_p + \|(y_n)_n\|_p$$

This is the triangle inequality for the norm $\|\cdot\|_p$ and the other properties of a norm are straightforward.

To show that it is complete (so a Banach space), we argue that every absolutely convergent series $\sum_k x_k$ in ℓ^p is convergent. To do that we show that the series of n^{th} terms must converge since $|x_{k,n}| \leq \|x_k\|_p$, define a sequence y by $y_n = \sum_k x_{k,n}$ and then show $y \in \ell^p$ and $\sum_k x_k = y$ in ℓ^p.

3. (a) [6 points] If (X, d) is a metric space, define the terms nowhere dense, first category and second category (for a subset of X). State the Baire category theorem.

Solution:

Definition 3.1. A subset $S \subset X$ of a metric space (X, d) (or topological space $(X, \mathscr{T})$ is called *nowhere dense* if the interior of its closure is empty, $(\bar{S})^\circ = \emptyset$.

A subset $E \subset X$ is called of *first category* if it is a countable union of nowhere dense subsets, that is, the union $E = \bigcup_{n=1}^\infty S_n$ of a sequence of nowhere dense sets S_n) $((\bar{S}_n)^\circ = \emptyset \forall n)$.

A subset $Y \subset X$ is called of *second category* if it fails to be of first category.

Theorem 3.2 (Baire Category). *Let (X, d) be a complete metric space which is not empty. Then the whole space $S = X$ is of second category in itself.*

(b) [7 points] Show that if (X, d) is a metric space with X countably infinite and if X has no isolated points, then (X, d) cannot be complete. [Hint: $x \in X$ is called isolated if $\{x\}$ is an open subset of X]

Solution: If $x \in X$, then $\{x\}$ is closed and since x is not an isolated point, $\overline{\{x\}}^\circ = \{x\}^\circ = \emptyset$ and so $\{x\}$ is nowhere dense in X. As X is countable $X = \bigcup_{x \in X}\{x\}$ is a countabel union of nowehere dnese subsets, hence X isof first category in itself. If (X, d) was complete, since X is not empty, the Baire cateogory theorem would contradict this. So (X, d) cannot be complete.

(c) [7 points] State the open mapping theorem and use it to show that a bijective bounded linear operator between Banach spaces must have bounded inverse.

Solution:

Theorem 3.3 (Open Mapping Theorem). *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces and $T \colon E \to F$ a surjective bounded linear operator. Then there exists $\delta > 0$ so that*

$$T(B_E) \supseteq \delta B_F$$

where $B_E = \{x \in E : \|x\|_E < 1\}$ and $B_F = \{y \in F : \|y\|_F < 1\}$ are the open unit balls of E and F.

Moreover, if $U \subseteq E$ is open, then $T(U)$ is open (in F).

Corollary 3.4. *If E, F are Banach spaces and $T \colon E \to F$ is a bounded linear operator that is also bijective, then T is an isomorphism.*

Proof. By the Open Mapping theorem T is automatically an open map, that is $U \subseteq E$ open implies $T(U) \subseteq F$ open. But, since T is a bijection the forward image $T(U)$ is the same as the inverse image $(T^{-1})^{-1}(U)$ of U under the inverse map T^{-1}.

Thus the open mapping condition says that T^{-1} is continuous. □

4. (a) [10 points] Let X_n be first countable topological spaces for $n = 1, 2, \ldots$ and let $x = (x_1, x_2, \ldots)$ be a point in the product $\prod_{n=1}^{\infty} X_n$ and let $\mathscr{B}_n = \{B_{n,1}, B_{n,2}, \ldots\}$ be a neighbourhood base at $x_n \in X_n$ for each n. Assume that $B_{n,1} \supseteq B_{n,2} \supseteq B_{n,3} \supseteq \cdots$ for each n.

Show that the sets of the form

$$B_{1,k} \times B_{2,k} \times \cdots \times B_{k,k} \times X_{k+1} \times X_{k+2} \times \cdots$$

make a countable neighbourhood base at x (in the product topology).

Solution: Let N be a neighbourhood of x. Then $x \in N^\circ$ with N° open in the product topology. So there is a basic open set B for the product topology with $x \in B \subseteq N^\circ \subseteq N$.

Basic open sets for the product topology are finite intersections

$$B = \pi_{j_1}^{-1}(U_1) \cap \pi_{j_2}^{-1}(U_2) \cap \pi_{j_\ell}^{-1}(U_\ell)$$

where $\ell \geq 0$, $j_1, j_2, \ldots, j_\ell$ are distinct elements indices $1 \leq j_i < \infty$ and $U_i \subseteq X_{j_i}$ is open for $1 \leq i \leq \ell$. As usual, we use π_j for the coordinate projection of the product space onto X_j. We can alternative write

$$B = \prod_{n=1}^{\infty} V_n$$

where

$$V_n = \begin{cases} U_i & \text{if } n = j_i \text{ for some } i, 1 \leq i \leq \ell \\ X_n & \text{otherwise.} \end{cases}$$

Alternatively we could write

$$B = V_1 \times V_2 \times \cdots \times V_r \times X_{r+1} \times X_{r+2} \times \cdots$$

where $r = \max\{j_i : 1 \leq i \leq \ell\}$.

From $x \in B$ we have $x_m \in V_m$ for $m = 1, 2, \ldots, r$.

Note that if $y = (y_1, y_2, \ldots) \in \prod_{n=1}^{\infty} X_n$ then $y \in B$ is equivalent to $y_m \in V_m$ for $1 \leq m \leq r$.

Since $x_m \in V_m$, V_m is open and $\mathscr{B}_m = \{B_{m,1}, B_{m,2}, \ldots\}$ is a neighbourhood base at $x_m \in X_m$, there must be $k_m \geq 1$ with $x_m \in B_{m,k_m} \subseteq V_m$. Since the sets in $\mathscr{B}_m$ are in decreasing order, we then also have $B_{m,k} \subseteq V_m$ for $k \geq k_m$.

Let $k_0 = \max\{k_1, k_2, \ldots, k_r\}$. Then

$$B_{1,k} \times B_{2,k} \times \cdots \times B_{r,k} \times X_{r+1} \times X_{r+2} \times \cdots \subseteq B$$

for $k > k_0$. So, if we also ensure that $k \geq r$ we have

$$B_{1,k}\times B_{2,k}\times\cdots\times B_{k,k}\times X_{k+1}\times X_{k+2}\times\cdots \subseteq B_{1,k}\times B_{2,k}\times\cdots\times B_{r,k}\times X_{r+1}\times X_{r+2}\times\cdots \subseteq B.$$

So we have the neighbourhood base property, but finally we need to check that the sets

$$B_{1,k} \times B_{2,k} \times \cdots \times B_{m,k} \times X_{k+1} \times X_{k+2} \times \cdots$$

are neighbourhoods of x. That is quite easy since $x_m \in B^\circ_{m,k}$ (for all m) and

$$B^\circ_{1,k} \times B^\circ_{2,k} \times \cdots \times B_{m,k} \circ \times X_{k+1} \times X_{k+2} \times \cdots$$

is an open set in the interior containing x.

(b) [10 points] Let $x_n = (x_{n,j})_{j=1}^\infty$ denote the sequence with j^{th} term

$$x_{n,j} = \begin{cases} \frac{1}{n} & \text{if } j = n \\ 0 & \text{for } j \neq n \end{cases}$$

Show that $\sum_{n=1}^\infty x_n$ is convergent in ℓ^p for $1 < p < \infty$ but fails to be absolutely convergent.

Solution: If we write out x_n in longhand, we find

$$x_n = (0, 0, \ldots, 0, \frac{1}{n}, 0, \ldots)$$

(or we could say that the sequence has all zero terms apart fron the n^{th} term, which is $1/n$). So if we compute for $1 \leq p < \infty$, we find

$$\|x_n\|_p = \left(\sum_{j=1}^\infty |x_{n,j}|^p\right)^{1/p} = \left(0^p + 0^p + \cdots + 0^p + \left(\frac{1}{n}\right)^p + 0^p + \cdots\right)^{1/p}$$

and so we get $\|x_n\|_p = (1/n^p)^{1/p} = 1/n$.

So

$$\sum_{n=1}^{\infty} \|x_n\|_p = \sum_{n=1}^{\infty} \frac{1}{n}$$

and this is infinite (harmonic series does not converge).

So the series $\sum_{n=1}^{\infty} x_n$ is not absolutely convergent in ℓ^p (no matter what p we choose).

The partial sums $S_n = x_1 + x_2 + \cdots + x_n$ work out as

$$\begin{aligned}
x_1 &= (1, 0, 0, \ldots) \\
x_2 &= (0, \frac{1}{2}, 0, 0, \ldots) \\
&\vdots \\
x_n &= (0, 0, \ldots, 0, \frac{1}{n}, 0, \ldots) \\
\hline
S_n &= (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)
\end{aligned}$$

It seems reasonable to guess that the limit of these S_n must be the sequence

$$S = (1, \frac{1}{2}, \frac{1}{3}, \ldots) = \left(\frac{1}{j}\right)_{j=1}^{\infty}$$

and we can see if that is so by looking at $\lim_{n\to\infty} \|S_n - S\|_p$. We have

$$S - S_n = (0, 0, \ldots, 0, -\frac{1}{n+1}, -\frac{1}{n+2}, \ldots),$$

aand for $1 < p < \infty$

$$\begin{aligned}
\|S_n - S\|_p &= \left(0^p + 0^p + \cdots 0^p + \left|-\frac{1}{n+1}\right|^p + \left|-\frac{1}{n+2}\right|^p + \cdots\right)^{1/p} \\
&= \left(\sum_{j=n+1}^{\infty} \frac{1}{j^p}\right)^{1/p}.
\end{aligned}$$

For $1 < p < \infty$, the series $\sum_{j=1}^{\infty} \frac{1}{j^p}$ converges, so that the tail sums $\sum_{j=n+1}^{\infty} \frac{1}{j^p} \to 0$ as $n \to \infty$.