Chapter 3: Baire category and open mapping theorems

A number of the major results rely on completeness via the Baire category theorem.

3.1 The Baire category theorem

3.1.1 Definition. A subset $S \subset X$ of a metric space $(X, d)$ (or topological space $(X, \mathcal{T})$ is called nowhere dense if the interior of its closure is empty, $(\bar{S})^\circ = \emptyset$.

A subset $E \subset X$ is called of first category if it is a countable union of nowhere dense subsets, that is, the union $E = \bigcup_{n=1}^{\infty} S_n$ of a sequence of nowhere dense sets $S_n$ ($(\bar{S}_n)^\circ = \emptyset \forall n$).

A subset $Y \subset X$ is called of second category if it fails to be of first category.

3.1.2 Example. (a) If a singleton subset $S = \{s\} \subset X$ (X metric) fails to be nowhere dense, then the interior of its closure is not empty. The closure $\bar{S} = S = \{s\}$ and if that has any interior it means it contains a ball of some positive radius $r > 0$. So

$$B_d(s, r) = \{x \in X : d(x, s) < r\} = \{s\}$$

and this means that $s$ is an isolated point of $X$ (no points closer to it than $r$).

An example where this is possible would be $X = \mathbb{Z}$ with the usual distance (so $B(n, 1) = \{n\}$) and $S$ any singleton subset. Another example is $X = B((2, 0), 1) \cup \{0\} \subset \mathbb{R}^2$ (with the distance on $X$ being the same as the usual distance between points in $\mathbb{R}^2$) and $S = \{0\}$.

(b) In many cases, there are no isolated points in $X$, and then a one point set is nowhere dense. So a countable subset is then of first category ($S = \{s_1, s_2, \ldots\}$ where the elements can be listed as a finite or infinite sequence).

For example $S = \mathbb{Z}$ is of first category as a subset of $\mathbb{R}$, though it of second category as a subset of itself. $S = \mathbb{Q}$ is of first category both in $\mathbb{R}$ and in itself (because it is countable and points are not isolated).
The idea is that first category means ‘small’ in some sense, while second category is ‘not small’ in the same sense. While it is often not hard to see that a set is of first category, it is harder to see that it fails to be of first category. One has to consider all possible ways of writing the set as a union of a sequence of subsets.

3.1.3 Theorem (Baire Category). Let \((X, d)\) be a complete metric space which is not empty. Then the whole space \(S = X\) is of second category in itself.

Proof. If not, then \(X\) is of first category and that means \(X = \bigcup_{n=1}^{\infty} S_n\) where each \(S_n\) is a nowhere dense subset \(S_n \subset X\) (with \((\overline{S_n})^c = \emptyset\)).

Since \(\overline{S_n}\) has empty interior, its complement is a dense open set. That is \(X \setminus \overline{S_n} = X \setminus (\overline{S_n})^c = X\).

Thus if we take any ball \(B_d(x, r)\) in \(X\), there is a point \(y \in (X \setminus \overline{S_n}) \cap B_d(x, r)\) and then because \(X \setminus \overline{S_n}\) is open there is a (smaller) \(\delta > 0\) with \(B_d(y, \delta) \subset (X \setminus \overline{S_n}) \cap B_d(x, r)\). In fact, making \(\delta > 0\) smaller again, there is \(\delta > 0\) with \(\overline{B_d(y, \delta)} \subset (X \setminus \overline{S_n}) \cap B_d(x, r)\).

Start with \(x_0 \in X\) any point and \(r_0 = 1\). Then, by the above reasoning there is a ball
\[
\overline{B_d(x_1, r_1)} = \{x \in X : d(x, x_1) \leq r_1\} \subset (X \setminus \overline{S_1}) \cap B_d(x_0, r_0),
\]
and \(r_1 < r_0/2 \leq 1/2\). We can then find \(x_2\) and \(r_2 \leq r_1/2 < 1/2^2\) so that
\[
\overline{B_d(x_2, r_2)} \subset (X \setminus \overline{S_2}) \cap B_d(x_1, r_1),
\]
and we can continue this process to select \(x_1, x_2, \ldots\) and \(r_1, r_2, \ldots\) with
\[
0 < r_n \leq r_{n-1}/2 < \frac{1}{2^n}, \quad \overline{B_d(x_n, r_n)} \subset (X \setminus \overline{S_n}) \cap B_d(x_{n-1}, r_{n-1}) \quad (n = 1, 2, \ldots)
\]

We claim the sequence \((x_n)_{n=1}^{\infty}\) is a Cauchy sequence in \(X\). This is because \(m \geq n \Rightarrow x_m \in B_d(x_n, r_n) \Rightarrow d(x_m, x_n) < r_n < 1/2^n\). So, if \(n, m\) are both large
\[
d(x_m, x_n) < \max\left(\frac{1}{2^n}, \frac{1}{2^m}\right)
\]
is small.

By completeness, \(x_\infty = \lim_{n \to \infty} x_n\) exists in \(X\). Since the closed ball \(\overline{B_d(x_n, r_n)}\) is a closed set in \(X\) and contains all \(x_m\) for \(m \geq n\), it follows that \(x_\infty \in \overline{B_d(x_n, r_n)}\).
for each \( n \). But \( \bar{B}_d(x_n, r_n) \subset X \setminus \bar{S}_n \) and so \( x_n \notin \bar{S}_n \). This is true for all \( n \) and so we have the contradiction

\[
x_n \notin \bigcup_{n=1}^{\infty} \bar{S}_n = X
\]

Thus \( X \) cannot be a union of a sequence of nowhere dense subsets. \( \square \)

3.1.4 Corollary. Let \( (X, d) \) be a compact metric space. Then the whole space \( S = X \) is of second category in itself.

Proof. Compact metric spaces are complete. So this follows from the theorem. \( \square \)

3.2 Open mapping theorem

3.2.1 Remark. We need to recall the definitions (2.5.2) of convergence of a series in normed space and of absolute convergence. Also recall from Proposition 2.5.3 that in a Banach space absolutely convergent series must converge — in fact that is not true for normed spaces that are incomplete.

3.2.2 Theorem (Open Mapping Theorem). Let \( (E, \| \cdot \|_E) \) and \( (F, \| \cdot \|_F) \) be Banach spaces and \( T: E \to F \) a surjective bounded linear operator. Then there exists \( \delta > 0 \) so that

\[
T(B_E) \supseteq \delta B_F
\]

where \( B_E = \{ x \in E : \| x \|_E < 1 \} \) and \( B_F = \{ y \in F : \| y \|_F < 1 \} \) are the open unit balls of \( E \) and \( F \).

Moreover, if \( U \subseteq E \) is open, then \( T(U) \) is open (in \( F \)).

Proof. First we show that the second statement (the ‘open mapping’ part) follows from the first.

It is handy to note that all (open) balls \( B(x_0, r) = \{ x \in E : \| x - x_0 \|_E < r \} \) in a normed space (or Banach space) \( E \) can be related to the unit ball \( B_E \) as follows

\[
B(x_0, r) = x_0 + rB_E
\]

(where \( rB_E = \{ rx : x \in B_E \} \) is a stretching of \( B_E \) be the factor \( r \) and \( x_0 + rB_E = \{ x_0 + rx : x \in B_E \} \) is the translate of \( rB_E \) by \( x_0 \)).
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So if $U \subseteq E$ is open and $y_0 \in T(U)$ then we know $y_0 = T(x_0)$ for some $x_0 \in U$. Since $U$ is open, there is $r > 0$ with $x_0 + rB_E = B(x_0, r) \subseteq U$. It follows from $T(B_E) \supseteq \delta B_F$ and linearity of $T$ that

$$T(U) \supseteq T(x_0 + rB_E) = T(x_0) + rT(B_E) \supseteq y_0 + r\delta B_F = B(y_0, r\delta).$$

Thus $y_0$ is an interior point of $T(U)$. As this is true of all $y_0 \in T(U)$, we have shown that $T(U)$ is open.

Now to prove the first assertion, let $V_r = T(rB_E) = rT(B_E)$ for $r > 0$. Because $T$ is surjective and $E = \bigcup_{n=1}^\infty nB_E$, we have

$$F = \bigcup_{n=1}^\infty T(nB_E) = \bigcup_{n=1}^\infty V_n.$$  

By the Baire category theorem (and this is where we use the assumption that $F$ is complete) we cannot have $V_n$ nowhere dense for all $n$. (See [5.1] for an explanation of what the Baire category theorem says, and its proof.) That means there is $n$ so that the closure $\bar{V}_n$ has nonempty interior, or in other words there is $y_0 \in \bar{V}_n$ and $r > 0$ with $B(y_0, r) = y_0 + rB_F \subseteq \bar{V}_n$.

By changing $y_0$ and reducing $r$ we can assume $y_0 \in V_n$ (and not just $y_0$ in the closure). The idea is that there must be $y_1 \in V_n \cap B(y_0, r/2)$ and then $B(y_1, r/2) \subseteq B(y_0, r) \subseteq \bar{V}_n$. Write $y_1 = T(x_0)$ for $x_0 \in nB_E$. Then we claim that $\bar{V}_{2n} \supseteq (r/2)B_F$. The argument is that $x_0 \in nB_E$ implies

$$2nB_E \supseteq nB_E - x_0$$

and so

$$V_{2n} \supseteq T(nB_E - x_0) = T(nB_E) - T(x_0) = V_n - y_1$$

Taking closures, we get

$$\bar{V}_{2n} \supseteq \bar{V}_n - y_1 \supseteq (y_1 + (r/2)B_F) - y_1 = (r/2)B_F.$$

The fact that the closure of $V_n - y_1$ is the same as $\bar{V}_n - y_1$ follows from the fact that the translation map $y \mapsto y - y_1: F \to F$ is distance preserving (and so a homeomorphism) of $F$ onto itself (with inverse the translation $y \mapsto y + y_1$). Homeomorphisms preserve all topological things, and map closures to closures.

For $k \in \mathbb{N}$ we have $\bar{V}_{kn} = (k\bar{V}_n) = k\bar{V}_n$ (one can check that because $y \mapsto ky$ is a homeomorphism with inverse $y \mapsto (1/k)y$). Thus, if $k$ big enough so that $kr/2 > 1$ we have

$$\bar{V}_{kn} = k\bar{V}_n \supseteq k\frac{r}{2}B_F \supseteq B_F.$$
Let $N = kn$. Thus we have $\bar{V}_N \supseteq B_F$. What we claim is that $V_{3N} \supseteq B_F$ (no closure now on the $V_{3N}$).

From $\bar{V}_N \supseteq B_F$ we deduce, for $j \in \mathbb{N}$,

$$\bar{V}_{N/j} = \frac{1}{j} V_N = \frac{1}{j} \bar{V}_N \supseteq \frac{1}{j} B_F$$

Starting with any $y \in B_F$ we must have $x_1 \in NB_E$ (so $T(x_1) \in V_N$) with

$$\|y - T(x_1)\|_F < \frac{1}{2}$$

or

$$y - T(x_1) \in \frac{1}{2} B_F \subseteq \bar{V}_{N/2}.$$

We can then find $x_2 \in (N/2) B_E$ (so $T(x_2) \in V_{N/2}$) with

$$\|(y - T(x_1)) - T(x_2)\|_F < \frac{1}{2^2}$$

or

$$y - T(x_1) - T(x_2) = y - (T(x_1) + T(x_2)) \in \frac{1}{2^2} B_F \subseteq \bar{V}_{N/2^2}.$$

By induction, we can find $x_1, x_2, \ldots$ with $x_j \in (N/2^{j-1}) B_E$ and

$$y - \sum_{i=1}^{j} T(x_i) \in \frac{1}{2^j} B_F \subseteq \bar{V}_{N/2^j}.$$

With this construction we have an absolutely convergent series $\sum_{j=1}^{\infty} x_j$ in $E$, because

$$\sum_{j=1}^{\infty} \|x_j\|_E \leq \sum_{j=1}^{\infty} \frac{N}{2^{j-1}} = 2N < 3N.$$

Now using completeness of $E$ (remember we used completeness of $F$ earlier) we know from Proposition 2.5.3 that $\sum_{j=1}^{\infty} x_j$ converges in $E$. That says there is $x \in E$ with $x = \lim_{j \to \infty} \sum_{i=1}^{j} x_i$ and in fact we know

$$\|x\|_E \leq \lim_{j \to \infty} \left\| \sum_{i=1}^{j} x_i \right\|_E \leq \lim_{j \to \infty} \sum_{i=1}^{j} \|x_i\|_E \leq 2N < 3N$$

because the closed ball $\bar{B}(0, 2N)$ is closed in $E$ and all the partial sums $\sum_{i=1}^{j} x_i$ are inside $\bar{B}(0, 2N)$. 
As $T$ is linear and continuous, we also have

$$Tx = \lim_{j \to \infty} T \left( \sum_{i=1}^{j} x_i \right) = \lim_{j \to \infty} \sum_{i=1}^{j} T(x_i) = y$$

because

$$\left\| y - \sum_{i=1}^{j} T(x_i) \right\|_F \leq \frac{1}{2j} \to 0$$

as $j \to \infty$.

So we have $y = T(x) \in V_{3N} = T(3NB_E)$. Since this is so for all $y \in B_F$, we have $B_F \subseteq V_{3N}$ as claimed. It follows that

$$T(B_E) \subseteq \frac{1}{3N} B_F,$$

which is the result with $\delta = 1/(3N) > 0$.

3.2.3 Corollary. If $E, F$ are Banach spaces and $T : E \to F$ is a bounded linear operator that is also bijective, then $T$ is an isomorphism.

Proof. By the Open Mapping theorem $T$ is automatically an open map, that is $U \subseteq E$ open implies $T(U) \subseteq F$ open. But, since $T$ is a bijection the forward image $T(U)$ is the same as the inverse image $(T^{-1})^{-1}(U)$ of $U$ under the inverse map $T^{-1}$.

Thus the open mapping condition says that $T^{-1}$ is continuous.

3.3 An application to Fourier series

Many details not given here.

If $f : [0, 1] \to \mathbb{C}$, then the Fourier series of $f$ is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi int}$$

(3.3.1)

where the Fourier coefficients are given by

$$\hat{f}(n) = \int_{[0,1]} f(t)e^{-2\pi int} \, dt$$

(3.3.2)
There are details missing from this. One is the restrictions on \( f \). Using the Lebesgue integral we would naturally impose the restriction that \( f \) is integrable, that is that \( f \) is measurable and

\[
\int_{[0,1]} |f(t)| \, dt < \infty
\]

Apart then from a discussion of almost everywhere equivalence classes of functions, that means that \( f \in L^1[0,1] \) is the natural place to consider Fourier series.

(If we restrict more to \( f \in L^2[0,1] \) we get to the situation where we are considering an orthonormal basis \( \{\phi_n\}_{n \in \mathbb{Z}} \) of the Hilbert space \( L^2[0,1] \) — with

\[
\phi_n(t) = e^{2\pi int}
\]

Then the Fourier series is an example of the expansion of a vector in a Hilbert space with respect to an orthonormal basis because

\[
\hat{f}(n) = \langle f, \phi_n \rangle.
\]

In this setting the general theory of orthonormal bases applies and quite a few strong consequences follow. However, one can argue that the \( L^1[0,1] \) setting is the natural generality to use.)

(This aside is premature anyhow as we have not discussed Hilbert spaces.)

Results that can be proved include:

(i) If \( f, g \in L^1[0,1] \) have the same Fourier series (meaning that if the coefficients agree, \( \hat{f}(n) = \hat{g}(n) \) for all \( n \in \mathbb{Z} \)) then \( f = g \) in \( L^1[0,1] \) (and recall that means that \( f \) and \( g \) agree almost everywhere).

(ii) (Riemann-Lebesgue Lemma) For \( f \in L^1[0,1] \),

\[
\lim_{|n| \to \infty} \hat{f}(n) = 0.
\]

(iii) Combining these two facts with some elementary ones, we can state it as that the map

\[
T: L^1[0,1] \to c_0
\]

given by

\[
Tf = (\hat{f}(0), \hat{f}(1), \hat{f}(-1), \hat{f}(2), \hat{f}(-2), \ldots) \quad (3.3.3)
\]

is a well-defined injective bounded linear operator.
Notice that the first statement says that the Fourier series of a function \( f \in L^1[0, 1] \) determines the function, though it does not say how to reconstruct the function.

The Riemann-Lebesgue Lemma needs some background information to prove it, principally that finite trigonometric polynomials, which take the form

\[
\sum_{n=-N}^{N} a_n e^{2\pi int}
\]

for \( 0 \leq N < \infty \) are dense in \( L^1[0, 1] \). It is easy to compute that the Fourier coefficients \( \hat{p}(n) \) for such a trigonometric polynomial \( p(t) = \sum_{n=-N}^{N} a_n e^{2\pi int} \) are \( \hat{p}(n) = a_n \) for \( -N \leq n \leq N \) and \( \hat{p}(n) = 0 \) for other \( n \). In particular note that \( Tp \in c_0 \) always.

It is also easy to see (using the triangle inequality) that

\[
|\hat{f}(n)| \leq \|f\|_1 = \int_{[0,1]} |f(t)| \, dt
\]

and so \( T \) maps \( L^1[0, 1] \) into \( \ell^\infty \). It is easy to check that \( T: L^1[0, 1] \to \ell^\infty \) is linear (\( T(f+g) = Tf + Tg \) and \( T(\lambda f) = \lambda Tf \) for \( f, g \in L^1[0, 1] \) and \( \lambda \in \mathbb{C} \)). Thus (3.3.4) gives that \( T: L^1[0, 1] \to \ell^\infty \) is a bounded linear operator mapping the dense subset of trigonometric polynomials into \( c_0 \), hence having range in \( c_0 \).

A natural question is to ask for a characterization of those series that have the right form to be Fourier series which are in fact the Fourier series of \( L^1[0, 1] \) functions. The first guess might be that the Riemann-Lebesgue Lemma tells the whole story, that \( T: L^1[0, 1] \to c_0 \).

3.3.1 Theorem. The range of the operator \( T: L^1[0, 1] \to c_0 \) defined by (3.3.3) is a proper subspace of \( c_0 \).

While it is possible to establish this result in several ways, a relatively painless way is to use the open mapping theorem. Since \( T \) is injective, if it was surjective then its inverse would be bounded. However, there is a ‘relatively’ simple way to contradict that possibility:

3.3.2 Lemma. Let \( D_N(t) = \sum_{n=-N}^{N} e^{2\pi int} \). Then \( \|T(D_N)\|_\infty = 1 \) but

\[
\lim_{N \to \infty} \|D_N\|_1 = \infty.
\]