MA3421 2016–17

Chapter 1: Bases for topologies

Contents

1.1 Review: metric and topological spaces

From MA2223 last year, you should know what a metric space is and what the metric topology is. Here is a quick refresher.

Definition 1.1.1. Given any set X of points and a function $d\colon X \times X \to [0, \infty) \subset \mathbb{R}$ with these 3 properties:

(i) $d(z, w) \geq 0$ with equality if and only if $z = w$;

(ii) $d(z, w) = d(w, z)$;

(iii) $d(z, w) \leq d(z, v) + d(v, w)$ (triangle inequality),

we say that d is a *metric on the space* X and we call the combination (X, d) a *metric space*.

Examples 1.1.2. $\mathbb{R}^n$ will denote the usual n-dimensional space (over $\mathbb{R}$) and $\mathbb{C}^n$ the complex version. We define the (standard) Euclidean distance between pairs of points in $\mathbb{R}^n$ by

$$d(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

(abstracting the distance formula from $\mathbb{R}^2$ or $\mathbb{R}^3$), for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. For $n = 1$, recall $d(x, y) = |x - y|$.

In the case on n-tuples $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, we also define the standard distance via

$$d(z, w) = \sqrt{\sum_{j=1}^{n} |z_j - w_j|^2}$$

Recall that for $z = x + iy \in \mathbb{C}$ (with $x, y \in \mathbb{R}$ the real and imaginary parts of z and $i^2 = -1$) the modulus (or absolute value) of such a z is $|z| = \sqrt{x^2 + y^2}$. *Properties:* $|z + w| \leq |z| + |w|$ (triangle inequality), $|zw| = |z|\,|w|$ ($z, w \in \mathbb{C}$).

The *complex conjugate* of $z = x + iy$ is $\bar{z} = x - iy$. *Properties:* $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$, $z\bar{z} = |z|^2$.

This gives us two examples $(\mathbb{R}^n, d)$ and $(\mathbb{C}^n, d)$ of metric spaces (for each $n \geq 1$).

Besides the 'natural' Euclidean metric on $\mathbb{R}^n$ or $\mathbb{C}^n$, there are several other very useful metrics (on the same set of points).

(a) $d_\infty(z, w) = \max_{1 \leq j \leq n} |z_j - w_j|$

(b) $d_1(z, w) = \sum_{1 \leq j \leq n} |z_j - w_j|$

(c) $d_p(z, w) = \left(\sum_{1 \leq j \leq n} |z_j - w_j|^p\right)^{1/p}$ for $1 \leq p < \infty$.

So d_1 and $d_2 = d$ are special cases of d_p and d_∞ is a kind of limiting case.

are examples. The first two are simple to check. (That is, it is quite easy to see that they are metrics, even easier than for the standard Euclidean one in fact.) The triangle inequality for d_p is called Minkowski's inequality and we should include a proof of it later.

So we also have the metric spaces $(\mathbb{R}^n, d_\infty)$, $(\mathbb{R}^n, d_p)$, $(\mathbb{C}^n, d_\infty)$ and $(\mathbb{C}^n, d_p)$ for $n \geq 2$, $1 \leq p < \infty$.

Notation 1.1.3. In any metric space (X, d) we define *open balls* as follows. Fix any point $x_0 \in X$ (which we think of as the centre) and any $r > 0$. Then the *open ball* of radius r centre x_0 is

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

The *closed ball* with the same centre and radius is

$$\bar{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}.$$

Definition 1.1.4 (Open sets in a metric space)**.** For a metric space (X, d) and a subset $G \subseteq X$ and a point $x \in G$, we say that x is *an interior point of* G if there is a ball $B(x, r)$ of some positive radius $r > 0$ centred at x so that $B(x, r) \subset G$.

A set $G \subseteq X$ is called *open* if each $x \in G$ is an interior point of G.

Remark 1.1.5. **Picture** for an open set: G contains **none** of its 'boundary' points.

In a metric space (X, d), any union $G = \bigcup_{i \in I} G_i$ of open sets $G_i \subseteq X$ is open (I any index set, arbitrarily large). Intersections $G_1 \cap G_2$ of two open sets $G_1, G_2 \subseteq X$ are also open.

Remark 1.1.6. In a metric space (X, d), we can talk about limits and convergence, and about continuous functions from X to $\mathbb{R}$ (or to $\mathbb{C}$, or to another metric space (Y, ρ)). We will recall how that is done below, but it is possible to express all these things using only the notion of an open subset, without referring to the actual distance d.

Of course the price is an increase in abstraction, but it shows clearly that many of the concepts don't really need the distances between points. As there are different metrics on X that give the same open sets, this additional abstraction shows that nothing about limits or continuity will be different if we change the actual metric in a way that does not change the open sets. One instance of that is provided by the examples $(\mathbb{R}^n, d)$ (with d the Euclidean distance), $(\mathbb{R}^n d_\infty)$, $(\mathbb{R}^n d_1)$ and more generally $(\mathbb{R}^n d_p)$ $(1 \leq p \leq \infty)$.

Definition 1.1.7. If X is a set then a *topology* $\mathscr{T}$ on X is a collection of subsets of X with the following properties

(i) $\phi \in \mathscr{T}$ and $X \in \mathscr{T}$;

(ii) if $U_i \in \mathscr{T}$ for all $i \in I =$ some index set, then $\bigcup_{i \in I} U_i \in \mathscr{T}$;

(iii) if $U_1, U_2 \in \mathscr{T}$, then $U_1 \cap U_2 \in \mathscr{T}$.

A set X together with a topology $\mathscr{T}$ on X is called a *topological space* $(X, \mathscr{T})$.

Remark 1.1.8. Normally, when we consider a topological space $(X, \mathscr{T})$, we refer to the subsets of X that are in $\mathscr{T}$ as *open subsets* of X.

We should perhaps explain immediately that if we start with a metric space (X, d) and if we take $\mathscr{T}$ to be the open subsets of (X, d) (according to the definition we gave earlier), then we get a topology $\mathscr{T}$ on X. We refer to that $\mathscr{T}$ as the *metric topology* on (X, d).

Example 1.1.9. (i) One example of a topology on any set X is the topology $\mathscr{T} = \mathcal{P}(X) =$ the power set of X (all subsets of X are in $\mathscr{T}$, all subsets declared to be open).

We can also get to this topology from a metric, where we define

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

In this metric the open ball of radius $1/2$ about any point $x_0 \in X$ is $B(x_0, 1/2) = \{x_0\}$ and all one point sets (singletons) are then open. As unions of open sets are open, it follows that all subsets are open.

The metric is called the discrete metric and the topology is called the discrete topology.

(ii) The other extreme is to take (say when X has at least 2 elements) $\mathscr{T} = \{\emptyset, X\}$. This is a valid topology, called the indiscrete topology.

If X has at least two points $x_1 \neq x_2$, there can be no metric on X that gives rise to this topology. If we thought for a moment we had such a metric d, we can take $r = d(x_1, x_2)/2$ and get an open ball $B(x_1, r)$ in X that contains x_1 but not x_2. As open balls in metric spaces are in fact open subsets, we must have $B(x_1, r)$ different from the empty set and different from X.

The only functions $f \colon X \to \mathbb{R}$ that are continuous are the constant functions in this example. On the other hand every function $g \colon Y \to X$ is continuous (no matter what Y is, as long as it is a topological space so that we can say what continuity means).

This example shows that there are topologies that do not come from metrics, or topological spaces where there is no metric around that would give the same idea of open set. Or, in other language, topological spaces that do not arise from metric spaces (are not metric spaces). Our example is not very convincing, however. It seems very silly, perhaps. But we will see some better examples later.

(iii) A more complicated way to get examples is to take subsets $T \subset X$ of a topological space $(X, \mathscr{T})$ and to define a topology on T by declaring $U \subset T$ to be open in case we can find $V \in \mathscr{T}$ with $T \cap V = U$.

In summary, we define a topology $\mathscr{T}_T$ on T by the rule

$$\mathscr{T}_T = \{T \cap V : V \in \mathscr{T}\}.$$

One can check that this gives a topology on T. The name usually given is the *subspace topology* (induced by the topology $\mathscr{T}$ on X).

In case the topology $\mathscr{T}$ on X came from some metric d on X, the subspace topology comes from the induced metric d_T (or submetric space structure) on T.

[To spell it out, what we mean is that $d_T \colon T \times T \to [0, \infty)$ is given by $d_T(t_1, t_2) = d(t_1, t_2)$. We use the distance d from X to get a distance d_T between points of T just by forgetting about distances to points of the complement $X \setminus T$. There are two obvious routes from (X, d) to a topology on T. One is to take the metric topology $\mathscr{T}$ on X and pass to $\mathscr{T}_T$. The other is to pass to the metric space (T, d_T) and from there to a metric topology on T. The idea is that both give the same topology $\mathscr{T}_T$.]

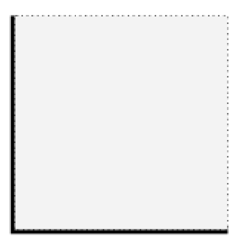
Definition 1.1.10. Let $(X, \mathscr{T})$ be a topological space.
A subset $F \subseteq X$ is called *closed* if its complement $X \setminus F$ is open.

Remark 1.1.11. **Picture** for a closed set: F contains **all** of its ‘boundary’ points.

Note that open and closed are opposite extremes. There are plenty of sets which are neither open nor closed.

One example is the ‘half-open interval’ $[0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ (which can also be called ‘half-closed’. The point 0 is not an interior point — so $[0, 1) \subset \mathbb{R}$ is not open. The point $1 \in \mathbb{R} \setminus [0, 1)$ is not an interior point of the complement. So the complement $\mathbb{R} \setminus [0, 1) = (-\infty, 0) \cup [1, \infty)$ is not open either. So $[0, 1) \subset \mathbb{R}$ is *neither* open *nor* closed. (Just to be careful, we should really say that we are using the standard absolute value metric on $\mathbb{R}$ and the corresponding metric topology — the usual topology to use for $\mathbb{R}$.)

An example that is perhaps more satisfying is $\{z = x + iy \in \mathbb{C} : 0 \leq x, y < 1\}$. It is a square in the plane $\mathbb{C} = \mathbb{R}^2$ with some of the ‘boundary’ included and some not. It is again neither open nor closed.



Proposition 1.1.12 (Simple properties of closed sets)**.** *Let $(X, \mathscr{T})$ be a topological space.*

(i) Any intersection $F = \bigcap_{i \in I} F_i$ of closed sets $F_i \subset X$ is closed.

(ii) Finite intersections of open sets are open.

(iii) Finite unions of closed sets are closed.

Proof. Exercise. [Use de Morgan’s laws for (i). Prove (ii) by induction from the case $n = 2$, which is in the definition of a topology. Then use de Morgan’s laws again for (iii).] $\square$

Definition 1.1.13. In a metric space (X, d), for any set $E \subseteq X$, the interior E° is the set of all its interior points. Combining with the definition of interior point, we have

$$E^\circ = \{x \in E : \exists r > 0 \text{ with } B(x, r) \subseteq E\}$$

Lemma 1.1.14. *Let (X, d) be a metric space and $E \subset X$. Then*

$$E^\circ = \bigcup \{G : G \subseteq E, G \textit{ open in } X\}$$

Definition 1.1.15. In a topological space $(X, \mathscr{T})$, for any set $E \subseteq X$, the interior E° is

$$E^\circ = \bigcup \{G : G \subseteq E, G \text{ open in } X\}$$

Lemma 1.1.16. *Let $(X, \mathscr{T})$ be a topological space and $E \subseteq X$. Then E° is the largest open subset of X contained in E.*

Proof. Exercise. □

Remark 1.1.17. **Picture:** E° is E minus all its 'boundary' points.

Definition 1.1.18. In a topological space $(X, \mathscr{T})$, for any set $E \subseteq X$, the *closure of* E is

$$\bar{E} = \bigcap \{F : F \subset X, E \subset F \text{ and } F \text{ closed}\}$$

Lemma 1.1.19. *Let $(X, \mathscr{T})$ be a topological space and $E \subseteq X$. Then $\bar{E}$ is the smallest closed subset of X containing E.*

Proof. Exercise. □

Remark 1.1.20. **Picture:** $\bar{E}$ is E with all its 'boundary' points added.

Proposition 1.1.21 (Properties of interiors and closures)**.** *Let $(X, \mathscr{T})$ be a topological space and $E \subseteq X$.*
$\bar{E} = X \setminus (X \setminus E)^\circ$ and $E^\circ = X \setminus (\overline{X \setminus E})$.

Proof. Exercise. □

Definition 1.1.22. Let $(X, \mathscr{T})$ be a topological space and $E \subseteq X$.
The *boundary* ∂E of E is defined as $\partial E = \bar{E} \setminus E^\circ$.

Remark 1.1.23. This formal definition makes the previous informal pictures into facts.

Remarks 1.1.24. The main properties of interiors, closures and boundaries are as follows (for $E \subset X$, X a topological space).

- (about complements) Proposition 1.1.21 above.
- $\partial(X \setminus E) = \partial E$.

- E is closed $\iff E = \bar{E} \iff \partial E \subseteq E$
- E is open $\iff E = E^\circ \iff \partial E \cap E = \emptyset$.
- (about unions of two sets $E_1, E_2 \subseteq X$) $E_1^\circ \cup E_2^\circ \subseteq (E_1 \cup E_2)^\circ$ and $\bar{E}_1 \cup \bar{E}_2 = \overline{E_1 \cup E_2}$.

  [The union of the interiors is not necessarily the interior of the union. Consider rational and irrational numbers in $\mathbb{R}$, which both have empty interior but where the union is open.]
- (about intersection of two sets $E_1, E_2 \subseteq X$) $\bar{E}_1 \cap \bar{E}_2 \supseteq \overline{E_1 \cap E_2}$ and $E_1^\circ \cap E_2^\circ = (E_1 \cap E_2)^\circ$.

  [The intersection of the closures can be a lot bigger than the closure of the intersection — look at rational and irrational numbers again.]

1.2 Review: continuity

Definition 1.2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces.

If $f\colon X \to Y$ is a function, then f is called *continuous at a point* $x_0 \in X$ if for each $\varepsilon > 0$ it is possible to find $\delta > 0$ so that

$$x \in X,\ d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$$

$f\colon X \to Y$ is called *continuous* if it is continuous at each point $x_0 \in X$.

If we have a function $g\colon T \to Y$ defined only on a subset $T \subset X$ we define continuity of g (on T) by thinking of T as a submetric space of X, and then requiring that g should be continuous in the above sense between (T, d_T) and (Y, d_Y).

Remark 1.2.2. It is also not hard to see that if we think of that range $Z = f(X)$ of a function $f\colon X \to Y$ (between metric spaces X and Y), then the function f is continuous if and only if it is continuous when we consider it as a function $f\colon X \to Z$ from X to the submetric space Z of Y.

Example 1.2.3. If (X, d) is a metric space and $f\colon X \to \mathbb{R}$ is a function, then when we say f is continuous we mean that it is continuous from the metric space X to the metric space $\mathbb{R} = (\mathbb{R}, \text{usual absolute value metric})$.

Similarly for complex valued functions $f\colon X \to \mathbb{C}$, we normally think of continuity to mean the situation where $\mathbb{C}$ has the usual metric.

Proposition 1.2.4. *If $f\colon X \to Y$ is a function between two metric spaces X and Y, then f is continuous if and only if it satisfies the following condition: for each open set $V \subset Y$, its inverse image $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.*

Proof. Exercise. □

Remark 1.2.5. Since the criterion in the Proposition is purely topological, we can use it to define continuity (on the whole space) of functions between topological spaces. We can be sure that this will be a generalisation of the metric case.

Definition 1.2.6. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f\colon X \to Y$ a function. We say that f is *continuous* if it satisfies

$$V \in \mathscr{T}_Y \Rightarrow f^{-1}(V) \in \mathscr{T}_X.$$

(In words, that says: for each open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X.)

Remark 1.2.7. We will come back later to the question of what it means for a function between topological spaces to be continuous at just one point.

Proposition 1.2.8. *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f\colon X \to Y$ a function. Let $Z = f(Y)$ and $\mathscr{T}_Z$ the topology induced on Z by $\mathscr{T}_Y$. Then f is continuous X to Y if and only if it is continuous when regarded as a function from X (with its topology $\mathscr{T}_X$) to Z with the topology $\mathscr{T}_Z$.*

Proof. Exercise. □

1.3 Review: limits of sequences

Definition 1.3.1 (Limits of sequence is a metric space)**.** Let (X, d) be a metric space (X, d).

A sequence $(x_n)_{n=1}^\infty$ in X is actually a function $x\colon \mathbb{N} \to X$ from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ to X where, by convention, we use the notation x_n instead of the usual function notation $x(n)$.

To say $\lim_{n\to\infty} x_n = \ell$ (with $\ell \in X$ also) means:

for each $\epsilon > 0$ it is possible to find $N \in \mathbb{N}$ so that

$$n \in \mathbb{N}, n > N \Rightarrow d(x_n, \ell) < \epsilon.$$

Remark 1.3.2. An important property of limits of sequences in metric spaces is that a sequence can have at most one limit. In a way we have almost implicitly assumed that by writing $\lim_{n\to\infty} x_n$ as though it is one thing. Notice however that there are sequences with no limit.

Proposition 1.3.3. *Let X and Y be two metric spaces. If $f\colon X \to Y$ is a function and $x_0 \in X$ is a point, then f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(z_0)$ holds for <u>all</u> sequences $(x_n)_{n=1}^\infty$ in X with $\lim_{n\to\infty} x_n = x_0$.*

Remark 1.3.4. Consider the case where we have sequences in $\mathbb{R}$, which is not only a metric space but where we can add and multiply.

One can show that the limit of a sum is the sum of the limits (provided the individual limits make sense). More symbolically,

$$\lim_{n\to\infty} x_n + y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Similarly

$$\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

if both individual limits exist.

We also have the result on limits of quotients,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

provided $\lim_{n\to\infty} y_n \neq 0$. In short the limit of a quotient is the quotient of the limits provided the limit in the denominator is not zero.

One can use these facts to show that sums and products of continuous $\mathbb{R}$-valued functions on metric spaces are continuous. Quotients also if no division by 0 occurs.

Definition 1.3.5 (Limits of sequences in a topological space)**.** Let $(X, \mathscr{T})$ be a topological space, $(x_n)_{n=1}^\infty$ a sequence of elements $x_n \in X$ and $\ell \in X$.

To say the sequence $(x_n)_{n=1}^\infty$ *converges to the limit* ℓ means:

for each $U \in \mathscr{T}$ with $\ell \in U$, there is $N \geq 1$ so that that

$$n \in \mathbb{N}, n > N \Rightarrow x_n \in U.$$

We use the notation "$x_n \to \ell$ as $n \to \infty$" to denote convergence to ℓ.

Lemma 1.3.6. *In the case of a metric space (X, d) and the corresponding metric topology $\mathscr{T}$, a sequence $(x_n)_{n=1}^\infty$ converges to a limit $\ell \in X$ in the topological space $(X, \mathscr{T})$ if and only if $\lim_{n\to\infty} x_n = \ell$ in (X, d).*

Proof. Exercise. □

Example 1.3.7. An important thing is that a sequence in a topological space may have more than one limit. We will come back to this, but here are two examples where it happens.

(i) Let $X = \{0, 1\}$ and $\mathscr{T}$ the indiscrete topology $\mathscr{T} = \{\emptyset, X\}$. Then every sequence $(x_n)_{n=1}^\infty$ converges to both 0 and 1.

Proof. The only open set that contains 0 is X, and $x_n \in X$ is always true. So $x_n \to 0$ as $n \to \infty$. Similarly $x_n \to 1$ as $n \to \infty$. □

(ii) Let $X = \{0, 1\}$ and $\mathscr{T} = \{\emptyset, \{0\}, X\}$. Then $(X, \mathscr{T})$ is a topological space. (Check!)

Consider the sequence $(x_n)_{n=1}^\infty$ where $x_n = 0$ for each n. Then $(x_n)_{n=1}^\infty$ has both 0 and 1 as limits.

Proof. To check that $\mathscr{T}$ is a topology, it is clear that $\emptyset, X \in \mathscr{T}$. Unions of sets from $\mathscr{T}$ are again in $\mathscr{T}$ (if all sets in the union are empty or if it is the empty union, the union is $\emptyset \in \mathscr{T}$; if not and X is one of the sets the union is $X \in \mathscr{T}$; finally if not either of the above all the sets in the union are contained in $\{0\}$ and $\{0\}$ is one of the sets in the union, so that the union is $\{0\} \in \mathscr{T}$).

Intersections of two sets from $\mathscr{T}$ are again in $\mathscr{T}$ (because for any two sets in $\mathscr{T}$ one is contained in the other and so the intersection of two sets from it is equal to one of the two sets).

The sequence $x_n = 0$ ($n = 1, 2, \ldots$) in X has limit 0. But it also has limit 1 because the only open sets that contains 1 is $X = \{0, 1\}$. So the constant zero sequence has two limits in this topology. □

Another strange fact is that every sequence in X converges to 1.

Definition 1.3.8 (Compactness)**.** Let $(X, \mathscr{T})$ be a topological space.

Let $T \subseteq X$. An *open cover* of T is a family $\mathcal{U}$ of open subsets of X such that

$$T \subseteq \bigcup\{U : U \in \mathcal{U}\}$$

A subfamily $\mathcal{V} \subseteq \mathcal{U}$ is called a subcover of $\mathcal{U}$ if $\mathcal{V}$ is also a cover of T.

T is called *compact* if each open cover of T has a finite subcover.

Remark 1.3.9. If (X, d) is a metric space, then a subset $T \subseteq X$ is called *bounded* if there exists $R \geq 0$ and $x_0 \in X$ with $T \subseteq \bar{B}(x_0, R)$.

One way to state the *Heine-Borel theorem* is that a subset $T \subseteq \mathbb{R}^n$ is compact if and only if it is both (1) closed and (2) bounded.

Proposition 1.3.10. *Continuous images of compact sets are compact: $T \subseteq X$ compact, $f : T \to Y$ continuous implies $f(T)$ compact.*

Definition 1.3.11 (sequential compactness)**.** If $(X, \mathscr{T})$ is a topological space, then a subset $T \subset X$ is called *sequentially compact* if it has the following property:

> Each sequence $(t_n)_{n=1}^\infty$ has a subsequence $(t_{n_j})_{j=1}^\infty$ that converges to some limit $\ell \in T$.

Theorem 1.3.12. *In a metric space (X, d) a subset $T \subset X$ is compact if and only if it is sequentially compact.*

Proof. Omitted here. □

Remark 1.3.13. It should be noted that this latter theorem and Proposition 1.3.3 do not hold for general topological spaces. We will return to this point later on.

We digress to mention connectedness very briefly because forward images (under continuous maps) of connected sets are connected, like forward images of compact sets. Most of the time it is inverse images that behave well.

Definition 1.3.14 (Connectedness)**.** We say a topological space $(X, \mathscr{T})$ is *connected* if the only subsets $S \subset X$ which are simultaneously open and closed are $S = \emptyset$ and $S = X$.

A subset $T \subset X$ is called a *connected subset* if it is connected when we consider it as a topological space by taking the subspace topology on it.

Proposition 1.3.15. *If $(X, \mathscr{T})$ is a topological space then X is connected if and only if it satisfies*

for each continuous $f\colon X \to \mathbb{R}$ with $f(X) \subseteq \{0,1\}$, f is constant.

Proposition 1.3.16. *Continuous images of connected sets are connected.*

That is, if X and Y are topological spaces, $f\colon X \to Y$ is continuous and if $T \subseteq X$ is connected, then $f(T) \subseteq Y$ is connected.

1.4 Base for a topology

In many cases, it is desirable to be able to describe all open sets in terms of simpler ones (as we do in a metric space when we use open balls to describe all open sets). This is the idea of the next definition.

Definition 1.4.1. If $(X, \mathscr{T})$ is a topological space, then a subfamily $\mathcal{B} \subseteq \mathscr{T}$ of the open sets is called a *base (for the open sets)* of the topology if

$$x \in U \subseteq X, U \text{ open} \Rightarrow \exists B \in \mathcal{B} \text{ with } x \in B \subset U$$

Example 1.4.2. (X, d) a metric space, $\mathcal{B} = \{B(x,r) : x \in X, r > 0\}$ (all open balls in X) is a base for the topology. This is true because of the way we define the metric topology (together with the fact that open balls $B(x,r)$ are open sets in the metric topology).

An example that is slightly more interesting is, in a metric space (X, d) again, $\mathcal{B} = \{B(x, 1/n) : x \in X, n \in \mathbb{N}\}$. To see that this is a base, consider $x \in U \subseteq X$ with U open. By definition of the metric topology, there is $r > 0$ with $B(x,r) \subseteq U$. But then there is $n \in \mathbb{N}$ with $1/n < r$, and for this n we have $x \in B(x, 1/n) \subseteq B(x,r) \subseteq U$. The fact $x \in B(x, 1/n) \subseteq U$ is what we wanted to establish, showing we have a base.

Proposition 1.4.3. *If $\mathcal{B}$ is a base for (the open sets of) a topology $\mathscr{T}$ on a set X, then every open set in X is a union of sets from $\mathcal{B}$.*

Proof. Let $U \subseteq X$ open (that is $U \in \mathscr{T}$). For each $x \in U$ there is $B \in \mathcal{B}$ with $x \in B \subseteq U$. Pick one such B for each x and call it B_x. So now we have a family $\{B_x : x \in U\}$ of basic open sets with $x \in U \Rightarrow x \in B_x \subseteq U$.

Consider the union $V = \bigcup_{x \in U} B_x$. We have $V \subseteq U$ since $B_x \subseteq U \forall x \in U$. But we have have $U \subseteq V$ since $x \in U \Rightarrow x \in B_x \subseteq V \Rightarrow x \in V$.

So $U = V =$ a union of basic open sets. □

Definition 1.4.4. A topological space $(X, \mathscr{T})$ is called *second countable* if there exists a base $\mathcal{B}$ for the topology where $\mathcal{B}$ is a countable collection of sets.

Example 1.4.5. Consider $X = \mathbb{R}^m$ ($m \geq 1$) with the standard Euclidean metric (and the metric topology). By $\mathbb{Q}^m$ we mean the points in $\mathbb{R}^m$ where all coordinates are rational numbers. Let $\mathcal{B} = \{B(q, 1/n) : q \in \mathbb{Q}^m, n \in \mathbb{N}\}$.

Then there is a surjection (in fact a bijection) : $\mathbb{Q}^m \times \mathbb{N} \to \mathcal{B}$ and, since $\mathbb{Q}^m \times \mathbb{N}$ is countable we have $\mathcal{B}$ countable.

To see that $\mathcal{B}$ is a base for the open sets in $\mathbb{R}^m$ consider $x \in U \subseteq \mathbb{R}^m$ with U open. Then there is an open ball $B(x,r) \subseteq U$ (a ball in the usual metric of radius $r > 0$). We can find $n \in \mathbb{N}$ with $1/n < r/2$. We can also find $q \in \mathbb{Q}^m$ with $d(x,q) < 1/n$ (because all m coordinates of x can be approximated by rational numbers within $1/(n\sqrt{m})$). Consider now $B(q,1/n)$.

As $d(x,q) < 1/n$ we have $x \in B(q,1/n)$. But we also have $B(q,1/n) \subseteq B(x,2/n)$ (because $y \in B(q,1/n) \Rightarrow d(y,q) < 1/n \Rightarrow d(y,x) \leq d(y,q) + d(q,x) < 1/n + 1/n = 2/n \Rightarrow y \in B(x,2/n)$). So we conclude $B(q,1/n) \subseteq B(x,2/n) \subseteq B(x,r) \subseteq U$.

In summary $x \in B(q,1/n) \subseteq U$ (for some $q \in \mathbb{Q}^m$ and $n \in \mathbb{N}$). This is what is needed to show we have a base (that we can do it for any $x \in U$ open).

We have a countable base and so we have shown $\mathbb{R}^m$ is second countable.

It can be quite useful to have a countable base, as certain constructions can then be done by considering only countably many open sets (rather than a much bigger number of all open sets as in $\mathbb{R}^m$).

We can generalise the construction in the example to other metric spaces if we can find a countable set to replace $\mathbb{Q}^m \subset \mathbb{R}^m$.

Definition 1.4.6. If $(X, \mathscr{T})$ is a topological space and $S \subseteq X$, then S is called *dense* in X if $\bar{S} = X$ (closure of S is X).

For example $S = \mathbb{Q}$ is dense in $X = \mathbb{R}$.

Definition 1.4.7. A topological space $(X, \mathscr{T})$ is called *separable* if there exists a countable subset S of X that is dense in X.

We usually say a 'countable dense subset'.

Example 1.4.8. If $m \geq 1$, then $\mathbb{Q}^m$ is a countable dense subset of $\mathbb{R}^m$ (using the usual topology).

On the other hand if X has the discrete topology, then the closure $\bar{S} = S$ for every $S \subset X$ (as every subset is open, and every subset is closed). So the only dense subset is $S = X$. It follows that a discrete space has to be countable if it is separable.

An uncountable set with the discrete topology is *not* separable.

Theorem 1.4.9. *Second countable topological spaces are always separable.*

Proof. Let $(X, \mathscr{T})$ be a second countable topological space. Let $\mathcal{B}$ be a countable base (for the open sets of) the topology. For each nonempty $B \in \mathcal{B}$, choose an element $x_B \in B$. Let $S = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$. [In fact, if the empty set is included in $\mathcal{B}$, we could remove the empty set from $\mathcal{B}$ and still have a countable base.]

Now S is a countable set (because the map $: \mathcal{B} \setminus \{\emptyset\} \to S : B \mapsto x_B$ is a surjective map from a countable set to S). We claim S is dense. If not $\bar{S} \neq X$ and there is $x \in X \setminus \bar{S}$. Since the complement of a closed set is open, there must be $B \in \mathcal{B}$ with $x \in B \subseteq X \setminus \bar{S}$. Now $B \neq \emptyset$ and so $x_B \in B \Rightarrow x_B \in X \setminus \bar{S}$. But $S \subseteq \bar{S} \Rightarrow X \setminus S \supseteq X \setminus \bar{S}$ and so we conclude $x_B \in X \setminus S$. But this contradicts $x_B \in S$. Hence S is countable dense.

Thus X is second countable. $\square$

Theorem 1.4.10. *Separable metric spaces are second countable.*

Proof. The proof for this is very much like the proof that $\mathbb{R}^m$ is second countable.

Let (X, d) be a metric space and $S \subseteq X$ a countable dense subset. We show that $\mathcal{B} = \{B(s, 1/n) : s \in S : n \in \mathbb{N}\}$ is a base via the same $1/n < r/2$ argument we used when $X = \mathbb{R}^m$ and $S = \mathbb{Q}^m$. □

This result does *not* hold for topological spaces in general. One way to produce an example is to start with a base for a (slightly strange) topology. We need to know what properties a base has to have.

Theorem 1.4.11. *Let X be a set.*

(i) If $\mathscr{T}$ is a topology on X and $\mathcal{B}$ is a base for (the open sets of) the topology, then we must have the following two properties satisfied by $\mathcal{B}$:

- $\bigcup\{B : B \in \mathcal{B}\} = X$
- $B_1, B_2 \in \mathcal{B}, x_0 \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B}$ *with* $x_0 \in B_3 \subseteq B_1 \cap B_2$.

(ii) Conversely, if $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of subsets of X that satisfies the above two properties, then there is a topology on X for which $\mathcal{B}$ is a base.

Proof. (i) Each base has the first property since X is open and open sets are unions of basic open sets (by Proposition 1.4.3).

If $x_0 \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is open and the existence of B_3 with $x_0 \in B_3 \subseteq B_1 \cap B_3$ follows by the definition of a base.

(ii) Assume $\mathcal{B}$ satisfies the two properties and take for $\mathscr{T}$ all possible unions of subcollections of $\mathcal{B}$. One detail is that we include the empty union as the empty set.

We claim that $\mathscr{T}$ is a topology on X and that $\mathcal{B}$ is a base for $\mathscr{T}$. We have a number of things to check to verify that $\mathscr{T}$ is a topology:

- $\emptyset \in \mathscr{T}$ — by our convention that we take the empty union to be included in $\mathscr{T}$
  $X \in \mathscr{T}$ — by the assumption that the union of $\mathcal{B}$ is X.
- $\mathscr{T}$ is closed under taking arbitrary unions of subcollections of $\mathscr{T}$ — true because a union of unions of sets from $\mathcal{B}$ is again a union of sets from $\mathcal{B}$.
- $\mathscr{T}$ is closed under taking intersections of two sets. If $U_1 = \bigcup_{i \in I_1} B_{1i}$ and $U_2 = \bigcup_{j \in I_2} B_{2j}$ (where I_1 and I_2 are index sets and $B_{1i} \in \mathcal{B} \forall i \in I_1$, $B_{2j} \in \mathcal{B} \forall j \in I_2$) then we have
  $$U_1 \cap U_2 = \bigcup_{i \in I_1, j \in I_2} B_{1i} \cap B_{2j}.$$
  Since we already know $\mathscr{T}$ is closed under taking arbitrary unions, we just need to know $B_{1i} \cap B_{2j} \in \mathscr{T} \forall i, j$.
  In other words it is sufficient to know $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathscr{T}$. For each $x \in B_1 \cap B_2$ we know, by the second assumption about $\mathcal{B}$ that there is $B_x \in \mathcal{B}$ with $x \in B_x \subseteq B_1 \cap B_2$. Then $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x \in \mathscr{T}$.

So this concludes the verification that $\mathscr{T}$ is a topology.

Finally $\mathcal{B}$ is a base for $\mathscr{T}$ because $\mathcal{B} \subseteq \mathscr{T}$ and $x \in U \in \mathscr{T} \Rightarrow \exists B \in \mathcal{B}$ with $x \in B \subseteq U$ because U is a union $\bigcup_{i \in I} B_i$ of sets $B_i \in \mathcal{B}$ and (at least) one of the B_i must contain x. This B_i will do for B.

□

Example 1.4.12. We introduce a *very unusual* topology on the set $\mathbb{R}$ or real numbers by taking as a base all intervals of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ (where $a < b$, $a, b \in \mathbb{R}$).

To see that there is a topology with this as its base, we just have to check the two simple properties of the theorem. As $x \in [x, x+1) \forall x \in \mathbb{R}$ we can see $\bigcup_{a<b}[a, b) = \mathbb{R}$. Next the intersection of two of these intervals $[a_1, b_1) \cap [a_2, b_2)$ is either empty or of the form $[a, b)$ where $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$. So if $x_0 \in [a_1, b_1) \cap [a_2, b_2)$, the intersection is not empty and there is $[a, b)$ with $x_0 \in [a, b) \subseteq [a_1, b_1) \cap [a_2, b_2)$.

The set of real numbers with this topology is called the *Sorgenfrey line*.

We claim that this topological space is separable but not second countable.

To see that it is separable, we show that the rationals are dense in this topological space. If not, there is a point x not in the closure of the rationals. Let Y denote this closure of the rationals and now we have $x \in \mathbb{R} \setminus Y$, an open set in the Sorgenfrey topology. Thus there is an interval $[a, b)$ with $a < b$ and $x \in [a, b) \subseteq \mathbb{R} \setminus Y$. But then there must be a rational $q \in [a, b) \cap \mathbb{Q}$. This $q \in [a, b) \subseteq \mathbb{R} \setminus Y \Rightarrow q \notin Y \Rightarrow q \notin \mathbb{Q}$ (as $\mathbb{Q} \subseteq Y =$ its closure). This is a contradiction. So $\mathbb{Q}$ is a dense subset (and countable). So the topology is separable.

It is not second countable because if $\mathcal{B}$ is any base for the open sets in this topology, then for each $x \in \mathbb{R}$ we have $x \in [x, x+1) =$ and open set in this space. So there must exist $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x+1)$. Notice then that x has to be the smallest element of B_x, thus guaranteeing that $\{B_x : x \in \mathbb{R}\}$ is an uncountable subset of $\mathcal{B}$. So there is no countable base for the topology.

Proposition 1.4.13. *Let $f \colon X \to Y$ be a function between topological spaces X and Y. Let $\mathcal{B}^Y$ be a base for the open sets in Y. Then the following are equivalent conditions:*

(a) f is continuous

(b) $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}^Y$.

Proof. (a) $\Rightarrow$ (b): This is clear since each $B \in \mathcal{B}^Y$ is open in Y.

(b) $\Rightarrow$ (a): Let $U \subseteq Y$ be open. Then we know $U = \bigcup_{i \in I} B_i$ for some collection $\{B_i : i \in I\}$ of basic open sets $B_i \in \mathcal{B}^Y$. So $f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B_i)$ is open in X (as a union of open sets). Thus f is continuous. □

1.5 Sub-bases, weak and product topologies

Remark 1.5.1. We consider now this question. Suppose we have a topological space X and we know that certain sets are open. Then what other sets must be open?

Definition 1.5.2 (Weaker and stronger topologies)**.** Let X be a set and suppose there are two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on X.

We say that $\mathscr{T}_1$ is *weaker* than $\mathscr{T}_2$ if there are fewer open sets in $\mathscr{T}_1$ than in $\mathscr{T}_2$, that is if $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

Sometimes we say that $\mathscr{T}_2$ is *stronger* than $\mathscr{T}_1$ to mean exactly the same thing, or $\mathscr{T}_2 \supseteq \mathscr{T}_1$.[1].

Theorem 1.5.3. *Given a set X and any collection $\mathscr{S}$ of subsets of X, there is a (unique) weakest topology $\mathscr{T}$ on X such that $\mathscr{S} \subseteq \mathscr{T}$. [In that case we call $\mathscr{S}$ a* sub-base *for the topology $\mathscr{T}$.]*

There are (at least) two ways to prove this, a slick way and a more informative way.

Proof (Slick way). Consider the all possible topologies on X that contain $\mathscr{S}$, that is consider

$$T_{\mathscr{S}} = \{\mathscr{U} : \mathscr{U}, \mathscr{U} \text{ a topology on } X, \mathscr{U} \supseteq \mathscr{S}\}.$$

We know there is at least one topology in $T_{\mathscr{S}}$ because the discrete topology $\mathcal{P}(X)$ is in $T_{\mathscr{S}}$.

Then we take

$$\mathscr{T} = \bigcap\{\mathscr{U} : \mathscr{U} \in T_{\mathscr{S}}\}$$

and verify that $\mathscr{T}$ is a topology. It then clearly satisfies $\mathscr{T} \subseteq \mathscr{U}$ for each $\mathscr{U} \in T_{\mathscr{S}}$ so that it is the weakest of all topologies on X that contain $\mathscr{S}$.

To verify that the intersection is a topology, note that $\phi, X \in \mathscr{T}$ since $T_{\mathscr{S}} \neq \emptyset$ and $\phi, X \in \mathscr{U}$ holds for each $\mathscr{U} \in T_{\mathscr{S}}$. For unions, suppose $U_i \in \mathscr{T}$ for each $i \in I$ (I some index set). Then $U_i \in \mathscr{U}$ for each $\mathscr{U} \in T_{\mathscr{S}}$. As $\mathscr{U}$ is a topology on X we have $\bigcup_{i\in I} U_i \in \mathscr{U}$. As this holds for each $\mathscr{U} \in T_{\mathscr{S}}$ we have $\bigcup_{i\in I} U_i \in \mathscr{T}$. The verification that $U_1, U_2 \in \mathscr{T}$ implies $U_1 \cap U_2 \in \mathscr{T}$ is similar. □

Proof (more constructive). We take $\mathcal{B}$ to be the set of all finite intersections of sets from $\mathscr{S}$ and show that $\mathcal{B}$ is a base for a topology on X.

To cover everything properly, we need to include the empty intersection, which we take to mean the whole space X. So the sets in $\mathcal{B}$ are all the sets of the form $S_1 \cap S_2 \cap \cdots \cap S_n$ where $n \geq 0$ and $S_1, S_2, \ldots, S_n \in \mathscr{S}$.

To show that $\mathcal{B}$ is a base for some topology $\mathscr{T}_{\mathcal{B}}$ on X we need to check the two properties from Theorem 1.4.11. Certainly $\bigcup\{B : B \in \mathcal{B}\} = X$ because $X \in \mathcal{B}$ and $B \subseteq X$ for each $B \in \mathcal{B}$. Also if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then we have $B_3 = B_1 \cap B_2 \in \mathcal{B}$ by the way we defined $\mathcal{B}$. So $x \in B_3 \subseteq B_1 \cap B_2$.

Finally, notice that any topology $\mathcal{U}$ on X that contains the sets in $\mathscr{S}$ must contain the finite intersections in $\mathcal{B}$. So it contains arbitrary unions of sets from $\mathcal{B}$. But these make up the topology $\mathscr{T}_{\mathcal{B}}$. So $\mathscr{T}_{\mathcal{B}} \subseteq \mathcal{U}$. Hence $\mathscr{T}_{\mathcal{B}}$ is the weakest topology in X containing $\mathscr{S}$. □

Theorem 1.5.4. *Let X be a set and $\{f_i \colon X \to Y_i : i \in I\}$ a collection of functions from X to topological spaces Y_i. Then there is a weakest topology on X that makes the maps f_i all continuous (called the 'weak topology generated by the maps f_i').*

---

[1] There is another fairly standard terminology of 'coarser' and 'finer', which we will avoid. Coarser means weaker and finer means the opposite inclusion.

A base for this weak topology is given by all finite intersections

$$\bigcap_{r=1}^{n} f_{i_r}^{-1}(U_r)$$

with $n \geq 0$, $i_1, i_2, \ldots, i_n \in I$ *distinct and* $U_r \subseteq Y_{i_r}$ *open.*

Proof. We want a topology $\mathscr{T}$ on X so that each f_i is continuous. That is we need $f_i^{-1}(U) \in \mathscr{T}$ for each open set $U \subseteq Y_i$.

We should take as our sub-base the family $\{f_i^{-1}(U) : i \in I, U \subseteq Y_i \text{ open}\}$. Then we can apply the above Theorem 1.5.3.

By the constructive proof of Theorem 1.5.3, a base for the weak topology can be obtained by taking arbitrary finite intersections of sets from the sub-base. That is all sets of the form

$$\bigcap_{r=1}^{n} f_{i_r}^{-1}(U_r)$$

with $n \geq 0$, $i_1, i_2, \ldots, i_n \in I$ and $U_r \subseteq Y_{i_r}$ open. We can eliminate repetitions among $i_1, i_2, \ldots, i_n$ because if $i_r = i_s$, then

$$f_{i_r}^{-1}(U_r) \cap f_{i_s}^{-1}(U_s) = f_{i_r}^{-1}(U_r) \cap f_{i_r}^{-1}(U_s) = f_{i_r}^{-1}(U_r \cap U_s)$$

and $U_r \cap U_s \subset Y_{i_r}$ is open. Thus we can eliminate one repetition by removing i_s and replacing U_r by $U_r \cap U_s$, and we can continue until all repetitions are gone. □

Definition 1.5.5. Let X and Y be two topological spaces. We define the topological product space to be the set $X \times Y$ (of ordered pairs (x, y) with $x \in X$, $y \in Y$) with the topology which has as its base

$$\{U \times V : U \subset X \text{ open}, V \subset Y \text{ open}\}$$

Remark 1.5.6. To justify the definition we must show that there is a topology on $X \times Y$ with the given base. Via Theorem 1.4.11 we just need to show that the union of all these sets in $X \times Y$ (easy as $X \times Y$ is one of them) and a property about intersections. As

$$(U_1 \times V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

is of the same form again ($U_1 \cap U_2$ is open in X if $U_1, U_2 \subset X$ open, and similarly for $V_1 \cap V_2$) the property about intersections is easy to verify.

So the definition is justified.

Theorem 1.5.7. *Let* X *and* Y *be topological spaces and* $\mathscr{T}_{prod}$ *the product topology on* $X \times Y$*. Denote by* $\pi_X \colon X \times Y \to X$, $\pi_Y \colon X \times Y \to Y$ *the coordinate projections given by* $\pi_X(x, y) = x$, $\pi_Y(x, y) = y$.

(i) $\pi_X \colon X \times Y \to X$ *and* $\pi_Y \colon X \times Y \to Y$ *are both continuous when* $X \times Y$ *is given the topology* $\mathscr{T}_{prod}$ *(and* X, Y *their original topologies).*

(ii) $\mathscr{T}_{prod}$ *is the weak topology on the set* $X \times Y$ *generated by the maps* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$.

(iii) If Z *is another topological space and* $f : Z \to X \times Y$ *is a function, then* f *is continuous if and only if* $\pi_X \circ f : Z \to X$ *and* $\pi_Y \circ f : Z \to Y$ *are each continuous.*

Proof. (i) If $U \subset X$ is open, then $(\pi_X)^{-1}(U) = U \times Y$ is open in $\mathscr{T}_{\text{prod}}$ (being a basic open set). Thus π_X is continuous.

As $(\pi_Y)^{-1}(V) = X \times V$ for $V \subset Y$, a similar argument shows that π_Y is continuous.

(ii) If there is a topology $\mathscr{T}'$ in which π_X and π_Y are each continuous, let $U \subset X$ and $V \subset Y$ be open.

As $(\pi_X)^{-1}(U) = U \times Y \in \mathscr{T}'$ follows by continuity of π_X and $(\pi_Y)^{-1}(V) = X \times V \in \mathscr{T}'$ by continuity of π_Y, we must have

$$U \times V = (U \times Y) \cap (X \times V) \in \mathscr{T}'.$$

As every set $W \in \mathscr{T}_{\text{prod}}$ is a union of basic open sets of the type $U \times V$ we have $W = \bigcup_{i \in I} U_i \times V_i \in \mathscr{T}'$.

Thus $\mathscr{T}_{\text{prod}} \subseteq \mathscr{T}'$.

(iii) To show that continuity of f implies continuity of both $\pi_X \circ f$ and $\pi_Y \circ f$, we use the fact that compositions of continuous functions are continuous. [Exercise: prove that.]

For the other direction of the $\iff$, assume now that both $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. Since $\pi_X \circ f$ is continuous, if $U \subset X$ is open then

$$(\pi_X \circ f)^{-1}(U) = f^{-1}((\pi_X)^{-1}(U)) = f^{-1}(U \times Y)$$

is open in Z. Similarly, if $V \subset Y$ is open, $(\pi_Y \circ f)^{-1}(V) = f^{-1}(X \times V)$ is open in Z. Thus

$$f^{-1}(U \times V) = f^{-1}((U \times Y) \cap (X \times V)) = f^{-1}(U \times Y) \cap f^{-1}(X \times V)$$

is open. As the inverse image of every basic open set in $X \times Y$ is open in Z, we conclude that f is continuous. (See Proposition 1.4.13 (b).)

□

Examples 1.5.8. Let $X = Y = \mathbb{R}$ with the usual topology (given by the absolute value metric $d(x_1, x_2) = |x_1 - x_2|$). Then there are several 'obvious' candidates for metrics on $X \times Y = \mathbb{R}^2$ which have been given before in Examples 1.1.2. They include

$$\begin{aligned} d_\infty((x_1, y_1), (x_2, y_2)) &= \max(|x_1 - x_2|, |y_1 - y_2|) \\ d_1((x_1, y_1), (x_2, y_2)) &= |x_1 - x_2| + |y_1 - y_2| \\ d((x_1, y_1), (x_2, y_2)) &= \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \end{aligned}$$

We know that each of these is a metric on $\mathbb{R}^2$.

Now, it turns out that the metric topology (open sets) will be the same in each of these metrics and will be the product topology. It is quite easy to see in the case of d_∞ that the metric topology is the product topology because open balls in d_∞ are products of open intervals:

$$\begin{aligned} B_{d_\infty}((x_0,y_0),r) &= \{(x,y)\in\mathbb{R}^2 : d_\infty((x,y),(x_0,y_0)) < r\} \\ &= (x_0-r,x_0+r)\times(y_0-r,y_0+r) \end{aligned}$$

and thus are basic open sets in the product topology. It follows that open sets in the metric topology of $\mathbb{R}^2, d_\infty)$ are open in the product topology. On the other hand if $W\subset\mathbb{R}^2$ is open in the product topology and $(x_0,y_0)\in W$, then there are open $U,V\subseteq\mathbb{R}$ with $(x_0,y_0)\in U\times V\subseteq W$. Thus there are $r_1,r_2>0$ so that

$$\{x\in\mathbb{R} : |x-x_0|<r_1\}\subset U, \{y\in\mathbb{R} : |y-y_0|<r_2\}\subset V.$$

For $r=\min(r_1,r_2)$ we have $B_{d_\infty}((x_0,y_0),r)\subseteq U\times V\subseteq W$ and so (x_0,y_0) is an interior point of W in the metric topology from $d^{(\infty)}$. Since $(x_0,y_0)\in W$ was arbitrary, W is open in the metric topology. This shows [in detail!] that the metric topology from d_∞ coincides with the product topology.

To show that all the metric topologies are the same, one way is to show that every open ball in one metric contains an open ball with the same centre (but a different radius). This shows that interior points of a set $W\subset\mathbb{R}^2$ are the same no matter which metric is used. If the notion of an interior point is the same, then so is the notion of open set.

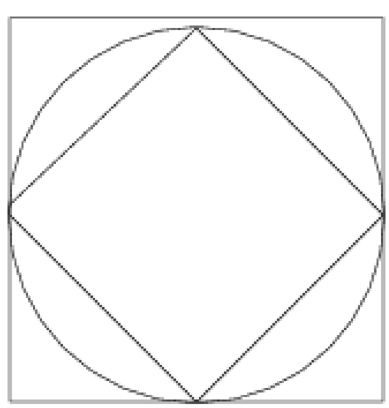
There are fairly elementary inequalities relating the 3 distances

$$d_\infty(p_1,p_2)\le d(p_1,p_2)\le d_1(p_1,p_2)\le 2d_\infty(p_1,p_2)$$

(for points $p_1,p_2\in\mathbb{R}^2$) and these show that

$$B_{d_1}(p,r)\subseteq B_d(p,r)\subseteq B_{d_\infty}(p,r)\subseteq B_{d_1}(p,2r)$$

It may help to draw these balls. $B_{d_1}(p,r)$ is a kind of diamond (of diagonal $2r$), B_d a disc (of radius r or diameter $2r$) and B_{d_∞} a square (of side $2r$).



Remark 1.5.9. A consequence of Theorem 1.5.7 (iii) (and the previous example) is that a function $f\colon Z \to \mathbb{R}^2$ on a topological space Z with values in the plane will be continuous if and only if the coordinate functions are continuous. In other words, if we write $f(\zeta) = (f_1(\zeta), f_2(\zeta))$ we know that f is continuous if and only if both $f_1\colon X \to \mathbb{R}$ and $f_2\colon X \to \mathbb{R}$ are continuous.

There is no such theorem in the other direction, for functions from $\mathbb{R}^2$ with values in some topological space. What you might guess could be true is that if (say) $F\colon \mathbb{R}^2 \to \mathbb{R}$ has $x \mapsto F(x, y_0)$ continuous for each fixed y_0 and also $y \mapsto F(x_0, y)$ continuous for each fixed x_0 then maybe F would be continuous on $\mathbb{R}^2$. The condition that fixing all but one of the variables in a multivariable function gives a continuous map is called *separate continuity*, but it does not imply continuity even in the simple case of two real variables. An example to show this is

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

It is easy to see that F is separately continuous but $F(x,x) = 1/2$ for all $x \neq 0$ while $F(0,0) = 0$ shows that F is not continuous at $(0,0) \in \mathbb{R}^2$ (by choosing x small we have $d((x,x),(0,0))$ as small as we want [for d the standard euclidean distance, say] but $F(x,x) - F(0,0) = 1/2$).

Definition 1.5.10. Consider a family $(X_i, \mathscr{T}_i)$ of topological spaces indexed by $i \in I$. Let $X = \prod_{i\in I} X_i$ be the cartesian product set and $\pi_j\colon X \to X_j$ the coordinate projection $\pi_j((x_i)_{i\in I}) = x_j$ (for $j \in I$).

Then (by Theorem 1.5.4) there is a (unique) weakest topology on X among all topologies $\mathscr{T}$ on X such that $\pi_i\colon (X, \mathscr{T}) \to (X_i \mathscr{T}_i)$ is continuous for each $i \in I$.

We call the weak topology on X generated by the family of coordinate projection $\pi_i\colon X \to X_i$ $(i \in I)$ the *product topology* on X.

Remark 1.5.11. Now notice from Theorem 1.5.4 that the collection of all such finite intersections

$$\mathcal{B} = \left\{ \bigcap_{r=1}^n (\pi_{i_r})^{-1}(U_r) : n \geq 0, i_1, i_2, \ldots, i_n \in I \text{ distinct}, U_r \in \mathscr{T}_{i_r} \right\}$$

forms a base for some topology on X. (We include the empty intersection $n = 0$ with the convention that the empty intersection means all of X, though this is not really necessary. As long as $I \neq \emptyset$ we can fix some $i \in I$ and consider $(\pi_i)^{-1}(X_i) = X$. This is covered by the $n = 1$ case and is in $\mathcal{B}$.)

The open sets in $\mathscr{T}_p$ are all unions of sets from $\mathcal{B}$.

Remarks 1.5.12. Notice that in the case of a product of two topological spaces $X_1 \times X_2$ (say) the product topology as defined in the statement of the foregoing theorem is the same as the product topology defined previously in 1.5.5 (because of Theorem 1.5.7 (ii)).

We can explain what the base $\mathcal{B}$ constructed above looks like in a way that is a bit more concrete. If $X = \prod_{i\in I} X_i$ as above, π_i also as above, and $V \subset X_i$ open, then $(\pi_i)^{-1}(V) = \prod_{j\in I} U_j$ where $U_j = X_j$ for all $j \neq i$ and $U_i = V$. Now, when we take an intersection of two such sets $(\pi_i)^{-1}(V)$ for different i.

Suppose $i_1 \neq i_2 \in I$ and $V_1 \subset X_{i_1}$, $V_2 \subset X_{i_2}$ both open. In that case we can write

$$(\pi_{i_1})^{-1}(V_1) \cap (\pi_{i_2})^{-1}(V_2) = \prod_{i \in I} U_i$$

where $U_i = X_i$ for $i \neq i_1, i \neq i_2$, and $U_{i_1} = V_1, U_{i_2} = V_2$.

If we carry out a similar argument for the finite intersections that make up $\mathcal{B}$, we find that each $B \in \mathcal{B}$ can be expressed as

$$B = \bigcap_{r=1}^{n} (\pi_{i_r})^{-1}(U_r)$$

where $n \geq 0$, $i_1, i_2, \ldots, i_n \in I$ are all different, $U_r \subset X_{i_r}$ open. We can then write

$$B = \prod_{i \in I} U_i$$

where $U_i = X_i$ for each $i \in I \setminus \{i_1, i_2, \ldots, i_n\}$ (all but finitely many $i \in I$) and $U_{i_r} = U_r$ for $1 \leq r \leq n$.

You could be tempted to define a base for a topology on the cartesian product X by taking all subsets of X of the form $\prod_{i \in I} U_i$ where $U_i \subset X_i$ open $\forall i \in I$, but this is a very different topology if I is infinite. Most of the nice properties that we will encounter about the product topology (such as Tychonoff's theorem on compactness of product spaces) would not be true if we used this more naive definition. The fact that the product topology is the weakest one making all π_i continuous is meant as a justification for the 'correct' (or standard) definition of the product topology.

Finally, a few other basic properties of the product topology.

Theorem 1.5.13. *Consider a family $(X_i, \mathscr{T}_i)$ of topological spaces indexed by $i \in I$. Let $X = \prod_{i \in I} X_i$ be the cartesian product set taken with the product topology, and $\pi_i \colon X \to X_i$ the coordinate projections ($i \in I$).*

(i) If Z is another topological space and $f \colon Z \to X$ is a function, then f is continuous if and only if $\pi_i \circ f \colon Z \to X_i$ is continuous for each $i \in I$.

[You could perhaps phrase this as saying that a function with values in a product space is continuous if and only if each of its coordinate functions is continuous.]

(ii) If $(x_n)_{n=1}^{\infty}$ is a sequence in X and $x_0 \in X$, then $x_n \to x_0$ as $n \to \infty$ if and only if $\pi_i(x_n) \to \pi_i(x_0)$ for each $i \in I$.

[You could perhaps phrase this as saying that a sequence in a product space converges if and only if it converges coordinate by coordinate.]

Proof. (i) If f is continuous, then so is the composition $\pi_i \circ f$.

On the other hand, if $\pi_i \circ f$ is continuous ($\forall i \in I$), then for $U_i \subset X_i$ open we have $(\pi_i \circ f)^{-1}(U_i) = f^{-1}((\pi_i)^{-1}(U_i))$ open in Z. As every basic open set for the product topology is a finite intersection $B = \bigcap_{r=1}^n (\pi_{i_r})^{-1}(U_r)$ (with $n \geq 0$, $i_1, i_2, \ldots, i_n \in I$, $U_r \in X_{i_r}$ open), we conclude

$$f^{-1}(B) = \bigcap_{r=1}^{n} f^{-1}((\pi_{i_r})^{-1}(U_r))$$

is open in Z (as a finite intersection of open sets). By Proposition 1.4.13 (b), f is continuous.

(ii) If $x_n \to x_0$, consider and open set $U_j \subset X_j$ with $\pi_j(x_0) \in U_j$. Since π_i is continuous (by the definition of the product topology), $U = \pi_j^{-1}(U_j)$ is open in the product topology, and $x_0 \in U$. Since we know $x_n \to x_0$, we know we can find $n_0 \in \mathbb{N}$ so that

$$x_n \in U = \pi_j^{-1}(U_j) \forall n \geq n_0.$$

We deduce

$$\pi_j(x_n) \in U_j \forall n \geq n_0.$$

This verifies $\pi_j(x_n) \to \pi_j(x_0)$ as $n \to \infty$.

Conversely, assume $\pi_i(x_n) \to \pi_i(x_0)$ as $n \to \infty$ holds for each $i \in I$. Let U be open (in the product topology) with $x_0 \in U$. Take a basic open set $B = \bigcap_{r=1}^n (\pi_{i_r})^{-1}(U_r)$ (with $n \geq 0$, $i_1, i_2, \ldots, i_n \in I$, $U_r \in X_{i_r}$ open) with $x_0 \in B \subseteq U$. Then $\pi_{i_r}(x_0) \in U_r$. Since $\pi_{i_r}(x_n) \to \pi_{i_r}(x_0)$ by assumption, there is $n_r \in \mathbb{N}$ so that $\pi_{i_r}(x_n) \in U_r \forall n \geq n_r$. Put $n_0 = \max_{1 \leq r \leq n} n_r$. Then

$$n \geq n_0 \Rightarrow x_n \in B = \bigcap_{r=1}^{n} (\pi_{i_r})^{-1}(U_r) \subseteq U.$$

We have shown $x_n \to x_0$ as $n \to \infty$.

□

1.6 Neighbourhood bases

In the proof of Theorem 1.4.10 we used the notion of distance in an essential way. Our next definition is designed to capture as much of the notion of 'close' as we can do in an arbitrary topological space. It is a useful notion, but perhaps we will see that it is limited as a substitute for what we can do in a metric space.

Definition 1.6.1. If X is a topological space, and $x_0 \in X$ is a point, then a subset $N \subseteq X$ is called a *neighbourhood* of the point x_0 if x_0 belongs to the interior of N.

We sometimes use the notation $\mathcal{U}_{x_0}$ of the collection of all neighbourhoods of x_0. The term *neighbourhood system* at x_0 is sometimes used to describe $\mathcal{U}_{x_0}$.

Remarks 1.6.2. We can see then that every superset of a neighbourhood of a point is again a neighbourhood of the same point. This means that neighbourhoods can be very irregular. For example, in the plane $\mathbb{R}^2$, neighbourhoods of the origin $(0,0)$ include balls $B((0,0),r)$ for $r>0$ and also closed balls $\bar{B}((0,0),r)$ or any set that contains some open ball about $(0,0)$. The positive x-axis union the ball (disc) of radius $1/4$ is an example of a neighbourhood. Possibly it is far-fetched to think that these points are all 'close' to $(0,0)$ in any reasonable sense.

In a topological space, we have no measure of closeness to a point x_0 and we take neighbourhoods as a substitute (no matter how strange the notation may be, it is quite useful).

An open set containing x_0 is always a neighbourhood of x_0. Thus every point has at least one neighbourhood — the whole space. In some cases, this can be the only neighbourhood of x_0.

The intersection $N_1 \cap N_2$ of any two neighbourhoods of x_0 is again a neighbourhood because $N_1^\circ \cap N_2^\circ = (N_1 \cap N_2)^\circ$. So $x_0 \in (N_1 \cap N_2)^\circ$ if $x_0 \in N_1^\circ$ and $x_0 \in N_2^\circ$.

Next we look at a concept that tries to cut down on the complication involved in thinking about all possible neighbourhoods of a point x_0.

Definition 1.6.3. If X is a topological space, and $x_0 \in X$ is a point, then a *neighbourhood base* at x_0 is a subset $\mathcal{B}_{x_0} \subset \mathcal{U}_{x_0}$ of the entire neighbourhood system $\mathcal{U}_{x_0}$ with the property

$$N \in \mathcal{U}_{x_0} \Rightarrow \exists B \in \mathcal{B}_{x_0} \text{ with } B \subseteq N$$

Examples 1.6.4. (i) In a metric space (X,d), for any point $x_0 \in X$

$$\mathcal{B}_{x_0} = \{B(x_0,r) : r > 0\}$$

(all the open balls centred at x_0) is a neighbourhood base at x_0.

If $N \in \mathcal{U}_{x_0}$, then $x_0 \in N^\circ$ and so there is $r>0$ with $B(x,r) \subseteq N^\circ \subseteq N$. Thus every neighbourhood contains an open ball centred at x_0. We also need to know that these balls are themselves neighbourhoods (which we do know).

(ii) So also is

$$\{B(x_0,1/n) : n \in \mathbb{N}\}$$

(balls of radius $1/n$).

(iii) In any topological space $\{U : U \text{ open}, x_0 \in U\}$ (all open sets containing x_0) is a neighbourhood base at x_0.

Definition 1.6.5. A topological space $(X,\mathscr{T})$ is called *first countable* at a point $x_0 \in X$ if there is a countable neighbourhood base at x_0.

We say the topological space is first countable if it is first countable at each of its points.

Proposition 1.6.6. *(i) Second countable topological spaces are always first countable.*

(ii) All metric spaces are first countable.

Proof. (i) If X is a second countable topological space, that means that there is a countable base $\mathcal{B}$ for the open sets of X. For a fixed $x_0 \in X$, take $\mathcal{B}_{x_0} = \{B \in \mathcal{B} : x_0 \in B\}$.

Clearly $\mathcal{B}_{x_0}$ (a subset of a countable set) must be countable. It is a neighbourhood base as, first it consists of neighbourhoods (open neighbourhoods) of x_0, and if N is any neighbourhood of x_0 then $x_0 \in N^\circ$ (open) $\Rightarrow \exists B \in \mathcal{B}$ (base for the open sets) with $x_o \in B \subseteq N^\circ \Rightarrow B \in \mathcal{B}_{x_0}$ and $B \subseteq N$.

(ii) We've seen that $\mathcal{B}_{x_0} = \{B(x_0, 1/n) : n \in \mathbb{N}\}$ is a neighbourhood base at any point x_0 of a metric space X. It is a countable neighbourhood base at x_0.

$\square$

Remark 1.6.7. It is tempting to wonder if the proof of Theorem 1.4.10 could be adapted to show that first countable and separable topological spaces have to be second countable, but it cannot. We can check that the Sorgenfrey line is first countable and separable but *not* second countable.

To see that the Sorgenfrey line is first countable, fix $x \in \mathbb{R}$ and note that $\mathcal{B}_x = \{[x, x+1/n) : n \in \mathbb{N}\}$ is a countable neighbourhood base at x. (To check that start with a neighbourhood N. Then x is in the interior N° and so there exist $a < b$ with $x \in [a, b) \subseteq N^\circ$. As $x < b$ there is $n \in \mathbb{N}$ with $1/n < b - x$. So $x \in [x, x + 1/n) \subseteq [a, b) \subseteq N^\circ \subseteq N$.

Although in this respect we cannot succeed, there are several cases where first countability is a simplifying assumption. Mostly it allows us to describe things via convergence of sequences.

Recall Definition 1.3.5 about convergence of sequences in a topological space.

Proposition 1.6.8. *(i) If $(x_n)_{n=1}^\infty$ a sequence in a topological space, $(X, \mathscr{T})$, $x_0 \in X$ and $\mathcal{B}_{x_0}$ a neighbourhood base at x_0. Then $x_n \to x_0$ as $n \to \infty \iff$ whenever $B \in \mathcal{B}_{x_0}$ there exists $n_0 \in \mathbb{N}$ with the property that*

$$x_n \in B \forall n \geq n_0.$$

(ii) If $(x_n)_{n=1}^\infty$ is a sequence in a metric space (X, d) and $x_0 \in X$, then $x_n \to x_0$ as $n \to \infty \iff$ the following $\varepsilon - n_0$ condition holds:

for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_0) < \varepsilon \forall n \geq n_0$$

Proof. (i) The $\Rightarrow$ direction is clear since basic neighbourhoods are neighbourhoods.

For $\Leftarrow$, assuming the condition about basic neighbourhoods, let $N \in \mathcal{U}_{x_0}$. Then there is a $B \in \mathcal{B}_{x_0}$ with $B \subseteq N$ (by the definition of a neighbourhood base). By the condition we are assuming there is $n_0 \in \mathbb{N}$ so that $n \geq n_0 \Rightarrow x_n \in B \Rightarrow x_n \in N$. So $x_n \in N \forall n \geq n_0$.

(ii) To deduce this from the previous part consider the neighbourhood base

$$\mathcal{B}_{x_0} = \{B(x_0, \varepsilon) : \varepsilon > 0\}$$

and observe that

$$x_n \in B(x_0, \varepsilon) \iff d(x_n, x_0) < \varepsilon.$$

$\square$

The second statement above is just saying that Definitions 1.3.1 and 1.3.5 agree for metric spaces, which we already stated in Lemma 1.3.6.

Proposition 1.6.9. *Let $(X, \mathscr{T})$ be a topological space and let $x \in X$ be a point, $\mathcal{U}_x$ the neighbourhoods of x. Then $N_1, N_2 \in \mathcal{U}_x \Rightarrow N_1 \cap N_2 \in \mathcal{U}_x$.*

[In words, the intersection of any two neighbourhoods of x is again a neighbourhood of x.]

Proof. If $N_1, N_2 \in \mathcal{U}_x$, then we have $x \in N_1^\circ$ and $x \in N_2^\circ$. Since $N_1^\circ \cap N_2^\circ$ is open [intersection of two open sets] and $x \in N_1^\circ \cap N_2^\circ \subseteq N_1 \cap N_2$ we have $x \in (N_1 \cap N_2)^\circ$, so that $N_1 \cap N_2 \in \mathcal{U}_x$.

[We could perhaps shorten the proof by using the fact noted in Remarks 1.1.24 that $(N_1 \cap N_2)^\circ = N_1^\circ \cap N_2$.] □

Examples 1.6.10. (i) Let X be an infinite set with the 'co-finite' topology. This means that the open sets are $\emptyset$ and subsets $U \subset X$ where $X \setminus U$ is finite. [It is not difficult to verify that this is a topology.]

If $(x_n)_{n=1}^\infty$ is a sequence in X where all the terms are distinct ($x_n \neq x_m$ if $n \neq m$), then $(x_n)_{n=1}^\infty$ has every point $x_0 \in X$ as a limit. (Exercise).

(ii) Let X be an uncountable set with the with the 'co-countable' topology. This means that the open sets are $\emptyset$ and subsets $U \subset X$ where $X \setminus U$ is countable. [It is not difficult to verify that this is a topology.]

If $(x_n)_{n=1}^\infty$ is a sequence in X and $x_n \to x_0$ as $n \to \infty$ (for some $x_0 \in X$), then there is $n_0 \in \mathbb{N}$ so that $x_n = x_0 \forall n \geq n_0$. Thus the sequence is eventually constant equal to its limit (and so the only convergent sequences are constant apart from some finite number of terms).

One way to think about this example is that very few sequences converge (and only to one limit). By contrast, in the previous example many sequences converged to very many limits.

We can characterise closures in some useful ways, and in a first countable space we can do it with limits of sequences.

Proposition 1.6.11. *Let $(X, \mathscr{T})$ be a topological space and $S \subseteq X$. Let $x_0 \in X$. Then the following are equivalent statements:*

(a) $x_0 \in \bar{S}$ (the closure of S);

(b) every open subset $U \in \mathscr{T}$ with $x_0 \in U$ has $U \cap S \neq \emptyset$;

(c) every neighbourhood $N \in \mathcal{U}_{x_0}$ has $N \cap S \neq \emptyset$;

(d) for some fixed neighbourhood base $\mathcal{B}_{x_0}$ at x_0, we have $B \cap S \neq \emptyset \forall B \in \mathcal{B}_{x_0}$.

(e) $X \setminus S \notin \mathcal{U}_{x_0}$

If X is assumed to be first countable, these are equivalent to *there exists a sequence $(s_n)_{n=1}^\infty$ with $s_n \in S \forall n \in \mathbb{N}$ so that $s_n \to x_0$ (in X) as $n \to \infty$.*

Proof. (a) $\Rightarrow$ (b): If, on the contrary $U \cap S = \emptyset$ for some $U \in \mathscr{T}$ with $x_0 \in U$, then $x_0 \in U \subset X \setminus S$. Hence $x_0 \in (X \setminus S)^\circ = X \setminus \bar{S}$ and $x_0 \notin \bar{S}$ — contradiction..

(b) $\Rightarrow$ (c): We have $x_0 \in N^\circ$ open. Hence $N^\circ \cap S \neq \emptyset$ and so $N \cap S \neq \emptyset$ (as $N^\circ \subseteq N$).

(c) $\Rightarrow$ (d): This is clear as basic neighbourhoods are neighbourhoods.

(d) $\Rightarrow$ (e): If $X \setminus S$ is a neighbourhood of x_0, then it contains some basic neighbourhood B. But $B \cap S \neq \emptyset$ by assumption. So $\emptyset = (X \setminus S) \cap S \supseteq B \cap S \neq \emptyset$ — a contradiction.

(e) $\Rightarrow$ (a): To show $x_0 \in \bar{S}$, assume not. Then $x_0 \in X \setminus \bar{S} =$ an open set, and $X \setminus \bar{S} = (X \setminus S)^\circ \subseteq X \setminus S$. So $X \setminus S$ is a neighbourhood of x_0, contradicting our assumption that (e) holds.

Finally, we show the equivalence of the sequence condition. First countability is needed in one direction only.

If there exists a sequence $(s_n)_{n=1}^\infty$ in S converging to x_0, we must have $x_0 \in \bar{S}$. If not, $N = X \setminus S$ is a neighbourhood of x_0, that is $x_0 \in N^\circ$ and so there is $n_0 \in \mathbb{N}$ with $s_n \in N^\circ \forall n \geq n_0$. But taking $n = n_0$ gives a contradiction $s_n = s_{n_0} \in S \cap N^\circ \subseteq S \cap N = S \cap (X \setminus S) = \emptyset$.

Take now X first countable and $x_0 \in \bar{S}$. We need to find a sequence in S that converges to S. Take a countable neighbourhood base $\mathcal{B}_{x_0} = \{B_1, B_2, B_3, \ldots\}$ at x_0. (In the eventuality that the base is actually finite, just keep repeating the last member to get a seemingly infinite neighbourhood base.) Pick $s_1 \in S \cap B_1$, $s_2 \in S \cap B_1 \cap B_2$ (possible since $B_1 \cap B_2 \in \mathcal{U}_{x_0}$), and so on. In general pick $s_n \in S \cap \bigcap_{j=1}^n B_j$. Clearly $(s_n)_{n=1}^\infty$ is now a sequence in S and we have to check it converges to x_0.

Take any open $U \subseteq X$ with $x_0 \in U$. Then $U \in \mathcal{U}_{x_0}$ and there is some basic neighbourhood $B_{n_0} \in \mathcal{B}_{x_0}$ with $B_{n_0} \subseteq U$. Observe now that $n \geq n_0 \Rightarrow s_n \in B_{n_0} \Rightarrow s_n \in U$. Thus we have n_0 so that $s_n \in U \forall n \geq n_0$. This establishes $s_n \to x_0$ as $n \to \infty$. □

Next we consider continuity of functions between topological spaces. Previously, in Definition 1.2.1, we defined continuity at a point and globally in the context of metric spaces. See also Proposition 1.3.3 (continuity via sequences). In Definition 1.2.6 we defined continuity *on the whole space only* (or global continuity) of functions on topological spaces.

We will now look into continuity a bit more fully in the context of topological spaces.

Definition 1.6.12. Let $f \colon X \to Y$ be a function between topological spaces X and Y. (We could say that $\mathscr{T}^X$ is the topology of X and $\mathscr{T}^Y$ the topology of Y, but we can get by without this notation.) Let $x_0 \in X$ and $y_0 = f(x_0) \in Y$.

By $\mathcal{U}_{x_0}^X$ we mean the neighbourhood system of x_0 in X and by $\mathcal{U}_{y_0}^Y$ we mean the neighbourhood system of y_0 in Y.

We say that f is *continuous at the point* $x_0 \in X$ if f satisfies

$$N^Y \in \mathcal{U}_{f(x_0)}^Y \Rightarrow f^{-1}(N^Y) \in \mathcal{U}_{x_0}^X.$$

Proposition 1.6.13. *Let $f \colon X \to Y$ be a function between topological spaces X and Y and fix $x_0 \in X$. Let $\mathcal{B}_{x_0}^X$ be a neighbourhood base at $x_0 \in X$ and $\mathcal{B}_{f(x_0)}^Y$ a neighbourhood base at $f(x_0) \in Y$. Then the following are equivalent conditions:*

(a) f *is continuous at* x_0*;*

(b) $\forall B^Y \in \mathcal{B}^Y_{f(x_0)}$ *we have* $f^{-1}(B^Y) \in \mathcal{U}^X_{x_0}$*;*

(c) $\forall B^Y \in \mathcal{B}^Y_{f(x_0)}$ *there exists* $B^X \in \mathcal{B}^X_{x_0}$ *with* $f(B^X) \subseteq B^Y$

If X *is first countable, then these are also equivalent to the sequence criterion:*

> *whenever* $(x_n)_{n=1}^\infty$ *is a sequence in* X *with* $x_n \to x_0$ *as* $n \to \infty$*, we have* $f(x_n) \to f(x_0)$ *as* $n \to \infty$*.*

Proof. (a) $\Rightarrow$ (b): This is clear since $\mathcal{B}^Y_{f(x_0)} \subseteq \mathcal{U}^Y_{f(x_0)}$.

(b) $\Rightarrow$ (c): This is clear since $f^{-1}(B^Y) \in \mathcal{U}^Y_{f(x_0)}$ implies there is $B^X \in \mathcal{B}^X_{x_0}$ with $B^X \subseteq f^{-1}(B^Y)$. This in turn can be restated $f(B^X) \subseteq B^Y$.

(c) $\Rightarrow$ (a): Let $N^Y \in \mathcal{U}^Y_{f(x_0)}$. Then there is $B^Y \in \mathcal{B}^Y_{f(x_0)}$ with $B^Y \subseteq N^Y$. By the assumption (c), there is $B_X \in \mathcal{B}^X_{x_0}$ with $f(B^X) \subseteq B^Y$. Hence $B^X \subseteq f^{-1}(B^Y) \subseteq f^{-1}(N^Y)$, from which we conclude $f^{-1}(N^Y) \in \mathcal{U}^X_{x_0}$.

This shows the equivalence of the 3 conditions.

Continuity at x_0 implies the sequence condition always. If f is continuous at x_0 and $x_n \to x_0$, to show $f(x_n) \to f(x_0)$, take any $U \subseteq Y$ open with $f(x_0) \in U$. Then $U \in \mathcal{U}^Y_{f(x_0)}$. Thus $N^X = f^{-1}(U) \in \mathcal{U}^X_{x_0}$. Since we know $x_n \to x_0$, we conclude we can find $n_0 \in \mathbb{N}$ so that

$$x_n \in (N_X)^\circ \subseteq N^X \forall n \geq n_0.$$

We deduce

$$f(x_n) \in f(N^X) \subseteq U \forall n \geq n_0.$$

So there is $n_0 \in \mathbb{N}$ satisfying $f(x_n) \in U \forall n \geq n_0$. This verifies $f(x_n) \to f(x_0)$ as $n \to \infty$. As this is true of every possible sequence $(x_n)_{n=1}^\infty$ in X converging to x_0, we have established that continuity implies the sequence criterion.

For the last implication, assume we are in a first countable space and that f satisfies the sequence criterion at x_0. Then there is a countable neighbourhood base $\mathcal{B}^X_{x_0} = \{B_1, B_2, \ldots\}$ at $x_0 \in X$.

To show continuity at x_0, select a neighbourhood $N^Y \in \mathcal{U}^Y_{f(x_0)}$. Let $V = f^{-1}(N^Y)$ and our claim is that V contains $B'_n = \bigcap_{j=1}^n B_j$ for some n. If not, then for each n, there is $x_n \in B'_n$ with $x_n \notin V$. This means that $f(x_n) \notin N^Y \forall n$.

We can verify that $x_n \to x_0$ (see the end of the proof of Proposition 1.6.11). We do not have $f(x_n) \to f(x_0)$ because it is not possible to find $n_0 \in \mathbb{N}$ with $f(x_n) \in N^Y \forall n \geq n_0$. (Note $f(x_{n_0}) \notin N^Y$ to see there is no such n_0.) As N^Y is a neighbourhood of $f(x_0)$ this shows $f(x_n) \to f(x_0)$ fails. So we have a sequence contradicting the sequence criterion we are assuming to hold.

We conclude from this that $\exists n$ with $B'_n \subseteq V$. As $B'_n \in \mathcal{U}^X_{x_0}$ we have $f^{-1}(N^Y) = V \in \mathcal{U}^X_{x_0}$. This establishes continuity of f at x_0. $\square$

Remark 1.6.14. Note that in the case where X and Y are metric spaces, we can take $\mathcal{B}^X_{x_0} = \{B(x_0, \delta) : \delta > 0\}$ and $\mathcal{B}^Y_{f(x_0)} = \{B(f(x_0), \varepsilon) : \varepsilon > 0\}$ and then the condition (c) above is equivalent to

$$\forall \varepsilon > 0 \text{ there exists } \delta > 0 \text{ so that } d^X(x, x_0) < \delta \Rightarrow d^Y(f(x), f(x_0)) < \varepsilon$$

See Definition 1.2.1.

Another remark is that first countability came into the proof when we needed to produce a sequence.

Proposition 1.6.15. *Let $f\colon X \to Y$ be a function between topological spaces X and Y. Then f is continuous (in the sense of definition 1.2.6) if and only if it is continuous at each point $x_0 \in X$.*

Proof. Suppose f is continuous (in the sense that $V \subset Y$ open in Y implies $f^{-1}(V) \subset X$ open in X). Let $x_0 \in X$ be a point and N^Y a neighbourhood of $f(x_0)$ in Y. So $V = (N^Y)^\circ$ is open in Y and $f(x_0) \in V$. Thus $x_0 \in f^{-1}(V) \subset f^{-1}(N^Y)$ and $f^{-1}(V)$ is open (in X). Thus x_0 is an interior point of $f^{-1}(N^Y)$, so that $f^{-1}(N^Y)$ is a neighbourhood of x_0. This shows f is continuous at x_0 (for each $x_0 \in X$).

Conversely, suppose f is continuous at each point $x_0 \in X$. Let $V \subset Y$ be open in Y. To show $f^{-1}(V) \subset X$ open in X, consider any point $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$ and V is a neighbourhood of $f(x_0)$ (open neighbourhood in fact). So by continuity of f at x_0, $f^{-1}(V)$ is a neighbourhood of x_0. So x_0 is an interior point of $f^{-1}(V)$. As this is true of each $x_0 \in f^{-1}(V)$, we see that $f^{-1}(V)$ is open. $\square$

Example 1.6.16. Let X_0 and X_1 be two disjoint uncountable sets and let $X = X_0 \cup X_1$ with the co-countable topology. Consider the function $f\colon X \to \mathbb{R}$ with $f(x) = 0$ for $x \in X_0$ and $f(x) = 1$ for $x \in X_1$.

Now f is not continuous because $f^{-1}((-1/2, 1/2)) = X_0$ is not open in X ($X \setminus X_0 = X_1$ is not countable).

However, f does satisfy the sequence criterion for continuity at each $x_0 \in X$ because $x_n \to x_0$ in X implies there is $n_0 \in \mathbb{N}$ so that $x_n = x_0 \forall n \geq n_0$. (See Examples 1.6.10 (ii).) Thus $f(x_n) \to f(x_0)$.

We conclude that *sequences do not describe continuity* in general topological spaces.

Neither do they describe closures because we can take $S = X_0$ in the same space X. Then the closure of S is the whole of X. (Reason: the only closed sets in X are the whole space X and countable sets. As S is uncountable, the only closed set that contains S is X.) But there are no sequences in S that converge to points of $X \setminus S$.

Remark 1.6.17. There is a (standard or well-known) idea of a *net*, a kind of generalisation of the notion of a sequence which addresses the issues raised above (and some more).

Oct 28/2016: Fixed typo in Theorem 1.5.4.
May 4, 2017: Fixed typo in Theorem 1.4.11.

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