MA2224 (Lebesgue integral) Tutorial/excercise sheet 8
[due April 6, 2018]

Name: Solutions

1. Use the Monotone Convergence Theorem to show that if $f\colon \mathbb{R} \to [0, \infty]$ (this should have been $f\colon \mathbb{R} \to [-\infty, \infty]$) is integrable, then

$$\int_{\mathbb{R}} f\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^+\,d\mu - \lim_{n\to\infty} \int_{[-n,n]} f^-\,d\mu$$

Solution: Since f is integrable we know that $\int_{\mathbb{R}} f^+\,d\mu < \infty$ and $\int_{\mathbb{R}} f^-\,d\mu < \infty$. Of course f^+ and f^- are measurable and by Q1 on sheet 7 (which was a fairly simple application of the Monotone Convergence Theorem),

$$\int_{\mathbb{R}} f^+\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^+\,d\mu \text{ and } \int_{\mathbb{R}} f^-\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^-\,d\mu$$

(since $f^+ \geq 0$ and $f^- \geq 0$). By definition then

$$\int_{\mathbb{R}} f\,d\mu = \int_{\mathbb{R}} f^+\,d\mu - \int_{\mathbb{R}} f^-\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^+\,d\mu - \lim_{n\to\infty} \int_{[-n,n]} f^-\,d\mu$$

2. Use the Monotone Convergence Theorem to show that if $f\colon \mathbb{R} \to [0, \infty]$ (this should have been $f\colon \mathbb{R} \to [-\infty, \infty]$) is measurable, and we have both

$$\lim_{n\to\infty} \int_{[-n,n]} f^+\,d\mu < \infty \text{ and } \lim_{n\to\infty} \int_{[-n,n]} f^-\,d\mu < \infty,$$

then f must be integrable.

Solution: By the Monotone convergence theorem applied to the increasing sequence $f_n = \chi_{[-n,n]} f^+$ (of nonnegative measurable functions) we have

$$\int_{\mathbb{R}} f^+\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]} f^+\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^+\,d\mu < \infty.$$

Similarly

$$\int_{\mathbb{R}} f^-\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]} f^-\,d\mu = \lim_{n\to\infty} \int_{[-n,n]} f^-\,d\mu < \infty.$$

Hence f is inetgrable (by definition).

3. Let $f(x) = e^{-x}\chi_{[0,\infty)}(x) - \frac{1}{1+x^2}\chi_{(-\infty,0)}(x)$. Show that f is integrable and compute $\int_{\mathbb{R}} f\,d\mu$. [Hint: $\chi_{[-n,0)}(x) = \chi_{[-n,0]}(x)$ for almost every $x \in \mathbb{R}$ and this implies $\int_{[-n,0)} 1/(1+x^2)\,d\mu(x) = \int_{[-n,0]} 1/(1+x^2)\,d\mu(x)$. More simply, we can get that from $\chi_{[-n,0]}(x)/(1+x^2) = \chi_{[-n,0)}(x)/(1+x^2) + \chi_{\{0\}}(x)$ and integrating both sides.]

Solution: In this case $f^+(x) = \max(f(x), 0) = e^{-x}\chi_{[0,\infty)}(x)$ and $f^-(x) = \max(-f(x), 0) = \frac{1}{1+x^2}\chi_{(-\infty,0)}(x)$.

We apply the previous question and that leads us to want to evaluate

$$\int_{[-n,n]} f^+\,d\mu \text{ and } \int_{[-n,n]} f^-\,d\mu.$$

We have

$$\int_{[-n,n]} f^+\,d\mu = \int_{\mathbb{R}} \chi_{[-n,n]} f^+\,d\mu = \int_{\mathbb{R}} \chi_{[-n,n]}(x) e^{-x}\chi_{[0,\infty)}(x)\,d\mu(x) = \int_{\mathbb{R}} \chi_{[0,n]}(x) e^{-x}\,d\mu.$$

By the theorem saying that the Lebesgue integral over a finite closed interval of a continuous function agrees with the Rieman integral,

$$\int_{\mathbb{R}} \chi_{[0,n]}(x) e^{-x}\,d\mu. = \int_0^n e^{-x}\,dx = [-e^{-x}]_0^n = -e^{-n} - (-e^0) = 1 - e^{-n}.$$

Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[-n,n]} f^+\,d\mu = 1.$$

Moving to f^-, we want

$$\int_{[-n,n]} f^-\,d\mu = \int_{\mathbb{R}} \chi_{[-n,n]} f^-\,d\mu = \int_{\mathbb{R}} \chi_{[-n,n]}(x) \frac{1}{1+x^2}\chi_{(-\infty,0)}(x)\,d\mu(x) = \int_{\mathbb{R}} \chi_{[-n,0)}(x) \frac{1}{1+x^2}\,d\mu(x)$$

By the hint this is the same as

$$\int_{\mathbb{R}} \chi_{[-n,0]}(x) \frac{1}{1+x^2}\,d\mu(x) = \int_{[-n,0]} \frac{1}{1+x^2}\,d\mu(x)$$

and then, as before, it equals the Riemann integral

$$\int_{-n}^0 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_{-n}^0 = \tan^{-1} 0 - \tan^{-1}(-n) = -\tan^{-1}(-n)$$

The limit as $n \to \infty$ of $\tan^{-1}(-n)$ is $-\pi/2$ and so

$$\int_{\mathbb{R}} f^-\,d\mu = \frac{\pi}{2} < \infty.$$

Finally, by the previous two questions,

$$\int_{\mathbb{R}} f\,d\mu = 1 - \frac{\pi}{2}.$$

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