MA2224 (Lebesgue integral) Tutorial sheet 7
[March 26, 2018]

Name: Solution

1. Use the Monotone Convergence Theorem to show that if $f \colon \mathbb{R} \to [0, \infty]$ is measurable, then
$$\int_{\mathbb{R}} f \, d\mu = \lim_{n \to \infty} \int_{[-n,n]} f \, d\mu$$

Solution: We know (by definition) that
$$\int_{[-n,n]} f \, d\mu = \int_{\mathbb{R}} f \chi_{[-n,n]} \, d\mu$$

If we take $f_n = f \chi_{[-n,n]}$ we find that f_n is measurable (product of two measurable fuctions), nonnegative (since $f \geq 0$ by assumption and $\chi_{[-n,n]} \geq 0$ because $\chi_{[-n,n]}$ has only the values 1 and 0) and $f_n \leq f_{n+1}$ always (because $\chi_{[-n,n]} \leq \chi_{[-n-1,n+1]}$ and $f \geq 0$).

Finally $\lim_{n \to \infty} f_n(x) = f(x)$ because $f_n(x) = f(x)$ once $n \geq |x|$ (when $\chi_{[-n,n]}(x) = 1$).

The Monotone Convergence Theorem says (under these circumstances) that‘
$$\int_{\mathbb{R}} f \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} f_n \, d\mu = \lim_{n \to \infty} \int_{[-n,n]} f \, d\mu$$

2. Using the previous question, find $\int_{\mathbb{R}} f \, d\mu$ for
$$f = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{[k,k+1)}$$

Solution: We can work out that
$$f \chi_{[-n,n]} = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{[k,k+1)} \chi_{[-n,n]} = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{[k,k+1) \cap [-n,n]} = \sum_{k=1}^{n-1} \frac{1}{2^k} \chi_{[k,k+1)} + \frac{1}{2^n} \chi_{\{n\}}$$

That is a simple function with integral
$$\int_{\mathbb{R}} f \chi_{[-n,n]} \, d\mu = \sum_{k=1}^{n-1} \frac{1}{2^k} \mu([k, k+1)) + \frac{1}{2^n} \mu(\{n\}) = \sum_{k=1}^{n-1} \frac{1}{2^k} + 0$$

Using the previous question, we have
$$\int_{\mathbb{R}} f \, d\mu = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1/2}{1 - 1/2} = 1$$

(because it is a geometric series with ratio $\frac{1}{2}$ and $|1/2| < 1$).

3. Let $f_n = \chi_{[n,2n]}$. Show that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in \mathbb{R}$ but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu \neq 0 = \int_{\mathbb{R}} 0 \, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x) \, d\mu(x).$$

   Solution: If $x \in \mathbb{R}$ is fixed, then for n large enough $x < n$ and so $f_n(x) = \chi_{[n,2n]}(x) = 0$. Thus $\lim_{n\to\infty} f_n(x) = 0$.

   We have

$$\int_{\mathbb{R}} f_n \, d\mu = \mu([n, 2n]) = 2n - n = n \not\to 0$$

   as $n \to \infty$.

4. Why does the previous question not conflict with the Monotone Convergence Theorem?

   Solution: The reason it does not is the we do not have monononicity of the seqence of functions.

   For instance $f_1 \leq f_2$ fails beecause $f_1(1) = 1 > 0 = f_2(1)$ and in general $f_n(n) = 1 > 0 = f_{n+1}(n)$.

   So we do **not have** $f_1 \leq f_2 \leq f_3 \leq \cdots$ (and moreover we do not start to have it eventually — that is, even if we left off a finite number of functions $f_1, f_2, \ldots, f_N$ from the sequence).

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