MA2224 (Lebesgue integral) Tutorial sheet 5
[March 12, 2018]

Name: Solutions

1. Verify directly that constant functions $f\colon \mathbb{R} \to (-\infty, \infty)$ are Lebesgue measurable.

   Solution: Let $c \in (-\infty, \infty)$ be the constant value so that $f\colon \mathbb{R} \to (-\infty, \infty)$ is given by $f(x) = c$ for each $x \in \mathbb{R}$.

   For $a \in \mathbb{R}$ we then have

   $$f^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \leq a\} = \begin{cases} \emptyset & \text{if } a < c \\ \mathbb{R} & \text{if } c \leq a \end{cases}$$

   Since we certainly have $\emptyset, \mathbb{R} \in \mathscr{L}$, we do have $f^{-1}((-\infty, a]) \in \mathscr{L}$ for each $a \in \mathbb{R}$ — that is we have f Lebesgue measurable.

2. If $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, show that $g \circ f$ is Lebesgue measurable. [Hint: If $E \subseteq \mathbb{R}$ is closed then $g^{-1}(E)$ is a closed set, hence a Borel set. Also $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$. What do we know about $f^{-1}(B)$ for B Borel?]

   Solution: The aim is to show that for each $a \in \mathbb{R}$ we have $(g \circ f)^{-1}((-\infty, a]) \in \mathscr{L}$.

   Taking $E = (-\infty, a]$ in the hint, we have

   $$(g \circ f)^{-1}((-\infty, a]) = f^{-1}(g^{-1}((-\infty, a]))$$

   Since g is continuous and $(-\infty, a] \subset \mathbb{R}$ is closed, we know that $g^{-1}((-\infty, a])$ is closed and hence Borel. (Recall that one way to characterise the Borel σ-algebra is that it is the σ-algebra generated by the closed sets.)

   Recall now the result that if f is Lebesgue measurable then $f^{-1}(B) \in \mathscr{L}$ holds for each Borel set B. So (taking $B = g^{-1}((-\infty, a])$)

   $$(g \circ f)^{-1}((-\infty, a]) = f^{-1}(g^{-1}((-\infty, a])) \in \mathscr{L}$$

   This is true for each $a \in \mathbb{R}$ and so f is Lebesgue measurable.

3. If $f\colon [0,1] \to \mathbb{R}$ is Lebesgue measurable on $[0,1]$ and $g(x) = e^{f(x)}/(1 + f(x)^2) - 1$, show that g is Lebesgue measurable on $[0,1]$.

   Solution: We defined (Lebesgue) measurability of functions defined on measurable subsets of $\mathbb{R}$ by extending the domain to $\mathbb{R}$ and giving the extension the value 0 on the complement. So, since f is measurable on $[0,1]$ it follows that the extension $F\colon \mathbb{R} \to \mathbb{R}$ given by

   $$F(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \\ 0 & \text{for } x \notin [0.1] \end{cases}$$

is measurable on $\mathbb{R}$.

Consider now the continuous function $h\colon \mathbb{R} \to \mathbb{R}$ given by

$$h(y) = \frac{e^y}{1+y^2} - 1$$

We have that $g(x) = h(f(x))$ for $x \in [0,1]$.

Because $h(0) = 0$ we have that the extension of g to $\mathbb{R}$ given by

$$G(x) = \begin{cases} g(x) & \text{if } x \in [0,1] \\ 0 & \text{for } x \notin [0.1] \end{cases}$$

is actually the same as $G = h \circ F$.

By the previous question G is measurable on $\mathbb{R}$ and so that means (by our definitions) that g is measurable on $[0,1]$, as required.

Some notes.

(i) One could mimic the previous question and prove that for functions $f\colon X \to \mathbb{R}$ defined on domains that are measurable subsets $X \subseteq \mathbb{R}$ such that f is measurable on X and for continuous $g\colon \mathbb{R} \to \mathbb{R}$, we do have $g \circ f$ measurable on X. What we would need for that is to show that if $B \subseteq \mathbb{R}$ is Borel, then $f^{-1}(B) \subseteq X$ is measurable. (That is a version of Proposition 3.2.4 of the notes.) Then the rest of the solution of Q2 would go through.

To do that we would consider $\Sigma_f = \{E \subseteq \mathbb{R} : f^{-1}(E) \in \mathscr{L}\}$ and show that Σ_f is a σ-algebra of subsets of $\mathbb{R}$ (in the same way as we did for the case $X = \mathbb{R}$ in Proposition 3.2.4 — one small detail is that now $f^{-1}(E^c) = X \setminus f^{-1}(E) = X \cap (f^{-1}(E))^c$ is the complement in X rather than in $\mathbb{R}$).

(ii) A device to avoid repeating those arguments is to consider $g_1(x) = g(x) - g(0)$ so that $g_1(0) = 0$ and $(g \circ f)(x) = (g_1 \circ f)(x) + g(0)$. The point is that if F is the extension

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{for } x \notin X \end{cases}$$

to domain $\mathbb{R}$, then $g_1 \circ F$ is the similar extension of $g_1 \circ f$. By Q2 then we know that (since g_1 is certainly continuous) $g_1 \circ F$ is measurable on $\mathbb{R}$. Then $g_1 \circ f$ is measurable on X.

Finally $g \circ f = g_1 \circ f + g(0)$ is measurable as the sum of a measurable function $g_1 \circ f$ and a (measurable) constant function.

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