MA2224 (Lebesgue integral) Tutorial sheet 3
[March 5, 2018]

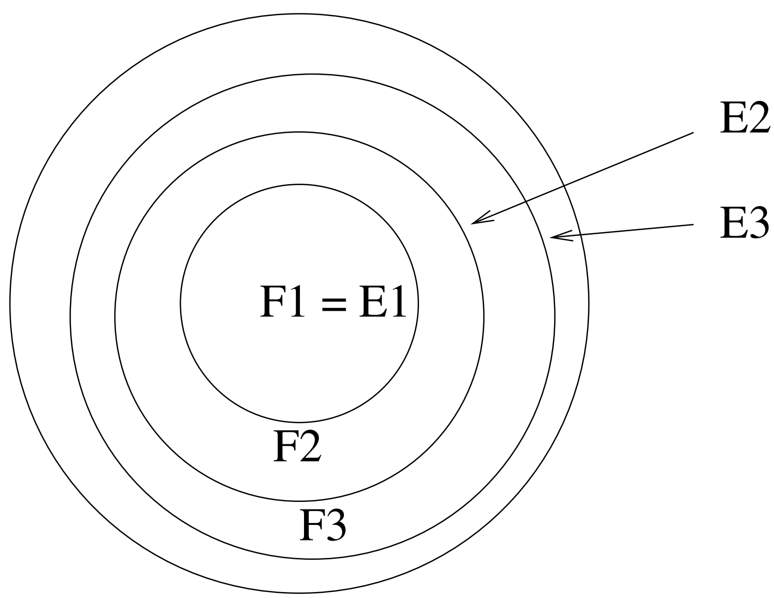
Name: Solutions

1. If $E_1, E_2, \ldots \in \mathscr{L}$ (the Lebesgue measurable sets) with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ (which means an increasing sequence of sets in $\mathscr{L}$), show that

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup_{n \geq 1} m^*(E_n).$$

[Hint: Consider $E_n \setminus E_{n-1}$ or $E_{n+1} \setminus E_n$ and make a disjoint sequence of sets with the same union as $\bigcup_{n=1}^{\infty} E_n$ and the union of the first n being E_n.]

Solution: The sequence $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_3 = E_3 \setminus E_2$, in general $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$, is disjoint (because $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ means that $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$ and so $F_n \cap E_i = \emptyset$ for $i < n$, and that gives $F_n \cap F_i = \emptyset$ as $F_i \subset E_i$). A Venn diagram should help:



Now

$$\bigcup_{i=1}^{n} F_i = E_n$$

for each n and

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{n=1}^{\infty} E_n.$$

By countable additivity of m^* (and disjointness of the F_i),

$$m^*\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m^*(F_i) = \sup_n \sum_{i=1}^{n} m^*(F_i) = \sup_n m^*\left(\bigcup_{i=1}^{n} F_i\right) = \sup_n m^*(E_n)$$

(We could perhaps explain that for series of non-negative terms, the partial sums are increasing and their limit is the same as their supremum. So we could add a step $\sum_{i=1}^{\infty} m^*(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} m^*(F_i) = \sup_n \sum_{i=1}^{n} m^*(F_i)$. Since $m^*(E_1) \le m^*(E_2) \le \cdots$ by monotonicity of m^* we can also write $\sup_n m^*(E_n) = \lim_{n\to\infty} m^*(E_n)$ to gibe an alternative version of the result.)

2. If $f_1, f_2, \ldots$ is a sequence of Lebesgue measurable functions $f_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) \le f_2(x) \le f_3(x) \le \cdots$$

holds for all $x \in \mathbb{R}$ and such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for all $x \in \mathbb{R}$, show that f must be a Lebesgue measurable function. [Hint: Because the sequence of functions is pointwise increasing, $f(x) \le a$ holds if and only if $f_n(x) \le a$ for all n.]

Solution: Our objective is to show that

$$f^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \le a\} \in \mathscr{L}$$

for each $a \in \mathbb{R}$.

Fix $a \in \mathbb{R}$.

Usng the hint

$$\{x \in \mathbb{R} : f(x) \le a\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{R} : f_n(x) \le a\}$$

Since f_n is Lebesgue measurable,

$$\{x \in \mathbb{R} : f_n(x) \le a\} \in \mathscr{L}$$

and so then is the (countable) intersection $\bigcap_{n=1}^{\infty} \{x \in \mathbb{R} : f_n(x) \le a\} \in \mathscr{L}$ (since σ algebras are closed under taking countable intersections).

3. Give an example of a function $f \colon \mathbb{R} \to \mathbb{R}$ that is not Lebesgue measurable. [Hint: is there a non-measurable set?]

Solution: We know there is a nonmeasurable subset of $\mathbb{R}$, that is $E \subseteq \mathbb{R}$ with $E \notin \mathscr{L}$.

Then $f = \chi_E$ is a nonmeasurable function since $f^{-1}((-\infty, 0]) = E^c = \mathbb{R} \setminus E \notin \mathscr{L}$.

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