MA2224 (Lebesgue integral) Tutorial sheet 3
[February 19, 2015]

Name: Solutions

1. Let $F = \bigcup_{n=1}^{\infty}(2n - (1/2^n), 2n]$. Explain why $F \in \mathscr{L}$ and compute $m^*(F)$ (based on results we have established).

   Solution: We know $(2n - (1/2^n), 2n] \in \mathscr{J}$ and $\mathscr{J} \subseteq \mathscr{L}$, so $(2n - (1/2^n), 2n] \in \mathscr{L}$ (for each n). We also know that $\mathscr{L}$ is a σ-algebra and so $F \in \mathscr{L}$.

   As m^* is countably additive on $\mathscr{L}$ and the sets $(2n - (1/2^n), 2n]$ are disjoint,

$$m^*(F) = \sum_{n=1}^{\infty} m^*((2n - (1/2^n), 2n]) = \sum_{n=1}^{\infty} m((2n - (1/2^n), 2n]) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

2. Show that $\mathbb{Q} \in \mathscr{L}$.

   Solution: We know that $\mathbb{Q}$ is countable (countably infinite in fact) and so we can list its elements $\mathbb{Q} = \{q_1, q_2, \ldots\}$. (If we like we can make sure there are no repetitions among the q_n.) So

$$\mathbb{Q} = \bigcup_{n=1}^{\infty}\{q_n\} = \bigcup_{n=1}^{\infty}[q_n q_n].$$

   As each $\{q_n\} = [q_n q_n] \in \mathscr{J} \subseteq \mathscr{L}$ and $\mathscr{L}$ is a σ-algebra, we have $\mathbb{Q} \in \mathscr{L}$.

   [A slightly different approach is to use countable subadditivity of m^* to say $m^*(\mathbb{Q}) \leq \sum_{n=1}^{\infty} m^*(\{q_n\}) = \sum_{n=1}^{\infty} m([q_n, q_n]) = 0$ and then use the fact that sets of outer measure zero are in $\mathscr{L}$.]

   [Both arguments can be adapted to show that every countable subset of $\mathbb{R}$ is a Lebesgue measurable set — nothing really special about $\mathbb{Q}$ except that it is infinite and finite sets are in $\mathscr{J}$ already.]

3. The 'usual' Cantor set (also called the middle thirds Cantor set) is constructed as an intersection $C = \bigcap_{n=1}^{\infty} C_n$ where $C_1 = [0, 1/3] \cup [2/3, 1]$ is obtained by removing the open middle third $(1/3, 2/3)$ from $[0, 1]$. Then each C_{n+1} is obtained form C_n by removing the open middle thirds from the (closed) intervals making up C_n. (For example $C_2 = [0, 1/3^2] \cup [2/3^2, 3/3^2] \cup [6/3^2, 7/3^2] \cup [8/3^2, 1]$.)

   Show that $C_n \in \mathcal{L}$ for each n and that $C \in \mathcal{L}$.

   Calculate $m^*(C_n)$ and $m^*(C)$.

   Solution: It is clear that C_n is a union of 2^n closed intervals, so that $C_n \in \mathscr{J} \subseteq \mathscr{L}$.

   Since $C \subseteq C_n$ for each n and m^* is monotone, we have $m^*(C) \leq m^*(C_n) = m(C_n)$. But we can see $m(C_1) = 1/3 + 1/3 = 2/3$ and by induction that $m(C_{n+1}) = (2/3)m(C_n)$

so that $m(C_n) = (2/3)^n$ for each $n \geq 1$. Thus $m^*(C) \leq (2/3)^n$ for each $n \geq 1$ and this forces $m^*(C) = 0$ (by letting $n \to \infty$).

As $\mathscr{L}$ is a σ-algebra

$$C = \bigcap_{n=1}^{\infty} C_n = \left(\bigcup_{m=1}^{\infty} C_n^c \right)^c \in \mathscr{L}.$$

[Note that from $m^*(C) = 0$ we get another argument that $C \in \mathscr{L}$, apart from the one using De Morgan's laws.]

Aside: One can show that C is an uncountable set. One way is to show that the elements $x \in C$ are exactly those that can be expressed in base 3 as

$$x = \sum_{n=1}^{\infty} \frac{d_n}{3^n}$$

where each 'digit' $d_n \in \{0, 2\}$. For instance $1/3 = 0/3 + \sum_{n=2}^{\infty} 2/3^n$, and so we have to use repeating 2's sometimes.

There are uncountably many possible sequences $d_1, d_2, \ldots$ of zeros and twos, and this is a way to show that C is uncountable.

So it is an uncountable set of measure zero (or total length zero) to add to countable examples like $\mathbb{Q}$.

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