MA2224 (Lebesgue integral) Tutorial sheet 2
[February 9, 2015]

Name: Solutions

1. For $E = (-2, 3] \cup (4, 7] \cup (8, 9] \cup (0, 4]$ compute $m(E)$ (where $m\colon \mathscr{J} \to [0, \infty]$ is length measure on the full interval algebra).

   Solution: In standard form $E = (-2, 7] \cup (8, 9]$ and so $m(E) = 7 - (-2) + 9 - 8 = 10$.

2. Let $\mathscr{J}$ be the (usual) full interval algebra (of subsets of $\mathbb{R}$) and define $\mu\colon \mathscr{J} \to [0, \infty]$ by

$$\mu(E) = \begin{cases} 0 & \text{if } m(E) < \infty \\ \infty & \text{if } m(E) = \infty \end{cases}$$

   [Aside: This will not be a useful way to do things.]

   Show that μ is finitely additive, monotone and finitely subadditive, but not countably additive or countably subadditive.

   Solution: μ monotone: if $E_1, E_2 \in \mathscr{J}$ with $E_1 \subseteq E_2$, we consider the two cases $m(E_1) < \infty$ and $m(E_1) = \infty$ separately. If $m(E_1) < \infty$, then $\mu(E_1) = 0 \leq \mu(E_2)$. If $m(E_1) = \infty$, then since m is monotone, $\infty = m(E_1) \leq m(E_2)$ and so $m(E_2) = \infty$ also. Thus $\infty = \mu(E_1) \leq \infty = \mu(E_2)$ holds in this case also.

   μ finitely additive: it is enough to show that if $E_1, E_2 \in \mathscr{J}$ are disjoint, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. If $m(E_1 \cup E_2) < \infty$, then $m(E_1) \leq m(E_1 \cup E_2) < \infty$ and similarly $m(E_2) < \infty$. So $\mu(E_1 \cup E_2) = 0 = \mu(E_1) + \mu(E_2)$.

   μ finitely subadditive: it is enough to show that if $E_1, E_2 \in \mathscr{J}$, then $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$.

   But by additivity and monotonicity (plus the fact that $\mathcal{J}$ is an algebra)

$$\mu(E_1 \cup E_2) = \mu(E_1 \cup (E_2 \setminus E_1)) = \mu(E_1) + m(E_2 \setminus E_1) \leq \mu(E_1) + \mu(E_2).$$

   μ not countably additive: If $E_n = [n, n+1)$, then $\bigcup_{n=1}^{\infty} E_n = [1, \infty) \in \mathcal{J}$ (and $E_n \in \mathcal{J}$ for each n. The sets E_n are pairwise disjoint and countable additivity of μ would imply

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

   But $\mu(E_n) = 0$ for each n while

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu([1, \infty)) = \infty.$$

   So countable additivity is false.

   μ not countably subadditive: The same example establishes that, as $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$ fails.

3. If $\mathcal{A}$ is an algebra of subsets of $\mathbb{R}$ and $\nu\colon \mathcal{A} \to [0, \infty]$ is a countably subadditive function such that $\nu(\emptyset) = 0$, show that it is finitely subadditive. [Recall: ν countably subadditive means that whenever $E \in \mathcal{A}$ and $E_1, E_2, E_3, \ldots$ is an infinite sequence of sets in $\mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^\infty E_n$, then $\nu(E) \leq \sum_{n=1}^\infty \nu(E_n)$. ν finitely subadditive means $\nu\left(\bigcup_{j=1}^n E_j\right) \leq \sum_{j=1}^n \nu(E_j)$ whenever $E_1, \ldots, E_n \in \mathcal{A}$.]

Solution: Starting with a finite list $E_1, \ldots, E_n \in \mathcal{A}$ we can extend it to an infinite list by taking $E_j = \emptyset$ for $j > n$.

Since $\mathcal{A}$ is an algebra, $E = \bigcup_{j=1}^n E_j \in \mathcal{A}$ and since the extra sets are all empty $E = \bigcup_{j=1}^\infty E_j$. By countable subadditivity we get

$$\nu(E) \leq \sum_{j=1}^\infty \nu(E_j) = \sum_{j=1}^n \nu(E_j)$$

since for $j > n$ we have $\nu(E_j) = \nu(\emptyset) = 0$.

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