MA2224 (Lebesgue integral) Tutorial sheet 1
[February 5, 2015]

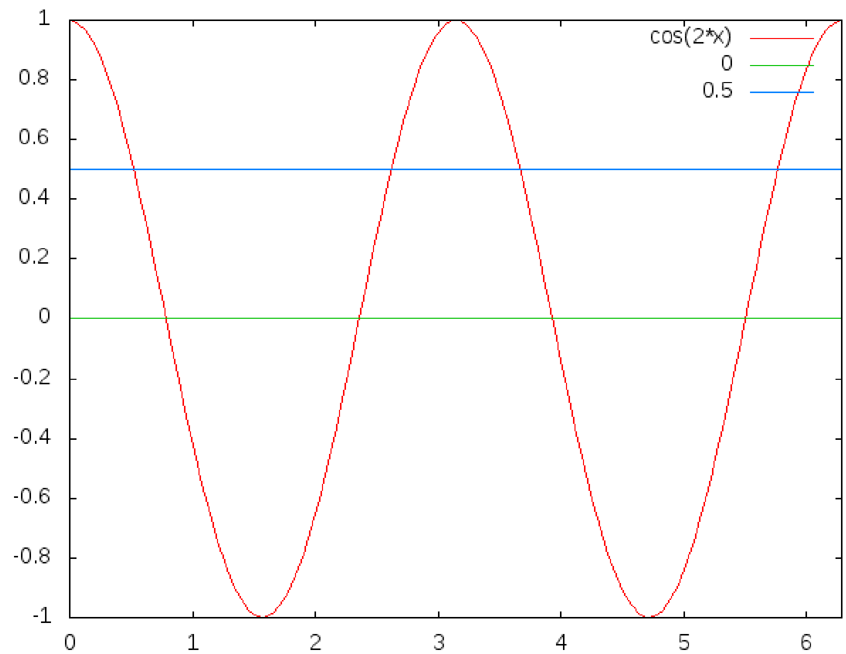
Name:

Student ID:

1. Let $f\colon [0, 2\pi] \to \mathbb{R}$ be given by $f(x) = \cos(2x)$. Find $f^{-1}([2, \infty))$ and $f^{-1}([0, 1/2])$.

   Solution: $f^{-1}([2, \infty)) = \{x \in [0, 2\pi] : f(x) \in [2, \infty)\} = \{x \in [0, 2\pi] : \cos(2x) \in [2, \infty)\} = \emptyset$ because $\cos(\theta) \in [-1, 1]$ always.

   For the other case $f^{-1}([0, 1/2])$, a graph helps

   We have $\cos(2x) = 1/2$ at $2x = \pi/3$, $2x = 2\pi - \pi/3$, $2x = 2\pi + \pi/3$ and $2x = 4\pi - \pi/3$, that is at $x = \pi/6$, $x = \pi - \pi/6$, $x = \pi + \pi/6$ and $x = 2\pi - \pi/6$.

   We have $\cos(2x) = 0$ at $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

   So
   $$f^{-1}([0, 1/2]) = \left[\frac{\pi}{6}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi - \frac{\pi}{6}\right] \cup \left[\pi + \frac{\pi}{6}, \frac{5\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi - \frac{\pi}{6}\right]$$

2. Give an example of $f\colon \mathbb{R} \to \mathbb{R}$ and two subsets $S_1, S_2 \subseteq \mathbb{R}$ where $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$.

   Solution: There are many possible examples but the function f must fail to be injective. If say $f(x) = 0$ (so that f is constant) the $S_1 = \{1\}$ and $S_2 = \{2\}$ will have $S_1 \cap S_2 = \emptyset$, $f(S_1 \cap S_2) = \emptyset$, $f(S_1) = \{0\} = f(S_2) = f(S_1) \cap f(S_2) \neq f(S_1 \cap S_2)$.

3. Let $\mathcal{A}$ denote the collection of all subsets of $\mathbb{R}$ that are either finite or complements of finite subsets. (Subsets with finite complement are called 'co-finite' subsets.) Show that $\mathcal{A}$ is an algebra.

   Solution:

   (a) $\emptyset \in \mathcal{A}$ since it is finite.

   (b) (Finite unions.) If $E_1, E_2 \in \mathcal{A}$, then we can consider two cases. In the first case both E_1 and E_2 are finite and then so is $E_1 \cup E_2$ finite — so $E_1 \cup E_2 \in \mathcal{A}$. In the second case at least one of E_1 or E_2 is infinite and then must have a finite complement. In that case
   $$(E_1 \cup E_2)^c = E_1^c \cap E_2^c$$
   must be finite (since at least one of E_1^c and E_2^c is finite). So $E_1 \cup E_2 \in \mathcal{A}$ (in all cases).

   (c) (complements.) If $E \in \mathcal{A}$, then either E is finite or E^c is finite. If E is finite then E^c is co-finite (has finite complement E) and so $E^c \in \mathcal{A}$ in that case. If E^c is finite, then also $E^c \in \mathcal{A}$.

Richard M. Timoney/J. Stalker