Chapter 4. The dominated convergence theorem and applications

The Monotone Covergence theorem is one of a number of key theorems alllowing one to exchange limits and [Lebesgue] integrals (or derivatives and integrals, as derivatives are also a sort of limit). Fatou's lemma and the dominated convergence theorem are other theorems in this vein, where monotonicity is not required but something else is needed in its place. In Fatou's lemma we get only an inequality for $\liminf$'s and non-negative integrands, while in the dominated convergence theorem we can manage integrands that change sign but we need a 'dominating' integrable function as well as existence of pointwise limits of the sequence of inetgrands.

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4.1 Fatou's Lemma

This deals with non-negative functions only but we get away from monotone sequences.

Theorem 4.1.1 (Fatou's Lemma)**.** *Let $f_n \colon \mathbb{R} \to [0, \infty]$ be (nonnegative) Lebesgue measurable functions. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu \geq \int_{\mathbb{R}} \liminf_{n\to\infty} f_n \, d\mu$$

Proof. Let $g_n(x) = \inf_{k\geq n} f_k(x)$ so that what we mean by $\liminf_{n\to\infty} f_n$ is the function with value at $x \in \mathbb{R}$ given by

$$\left(\liminf_{n\to\infty} f_n\right)(x) = \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\inf_{k\geq n} f_k(x)\right) = \lim_{n\to\infty} g_n(x)$$

Notice that $g_n(x) = \inf_{k\geq n} f_k(x) \leq \inf_{k\geq n+1} f_k(x) = g_{n+1}(x)$ so that the sequence $(g_n(x))_{n=1}^{\infty}$ is monotone increasing for each x and so the Monotone convergence theorem says that

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n \, dx = \int_{\mathbb{R}} \lim_{n\to\infty} g_n \, d\mu = \int_{\mathbb{R}} \liminf_{n\to\infty} f_n \, d\mu$$

But also $g_n(x) \leq f_k(x)$ for each $k \geq n$ and so

$$\int_{\mathbb{R}} g_n \, d\mu \leq \int_{\mathbb{R}} f_k \, d\mu \qquad (k \geq n)$$

or

$$\int_{\mathbb{R}} g_n \, d\mu \leq \inf_{k \geq n} \int_{\mathbb{R}} f_k \, d\mu$$

Hence

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \lim_{n\to\infty} \left(\inf_{k\geq n} \int_{\mathbb{R}} f_k \, d\mu \right) \geq \lim_{n\to\infty} \int_{\mathbb{R}} g_n \, d\mu = \int_{\mathbb{R}} \liminf_{n\to\infty} f_n \, d\mu$$

□

Example 4.1.2. Fatou's lemma is not true with 'equals'.

For instance take, $f_n = \chi_{[n,2n]}$ and notice that $\int_{\mathbb{R}} f_n \, d\mu = n \to \infty$ as $n \to \infty$ but for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 0$. So

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \infty > \int_{\mathbb{R}} \liminf_{n\to\infty} f_n \, d\mu = \int_{\mathbb{R}} 0 \, d\mu = 0$$

This also shows that the Monotone Convergence Theorem is not true without 'Monotone'.

4.2 Almost everywhere

Definition 4.2.1. We say that a property about real numbers x holds *almost everywhere* (with respect to Lebesgue measure μ) if the set of x where it fails to be true has μ measure 0.

Proposition 4.2.2. *If $f \colon \mathbb{R} \to [-\infty, \infty]$ is integrable, then $f(x) \in \mathbb{R}$ holds almost everywhere (or, equivalently, $|f(x)| < \infty$ almost everywhere).*

Proof. Let $E = \{x : |f(x)| = \infty\}$. What we want to do is show that $\mu(E) = 0$.

We know $\int_{\mathbb{R}} |f| \, d\mu < \infty$. So, for any $n \in \mathbb{N}$, the simple function $n\chi_E$ satisfies $n\chi_E(x) \leq |f(x)|$ always, and so has

$$\int_{\mathbb{R}} n\chi_E \, d\mu = n\mu(E) \leq \int_{\mathbb{R}} |f| \, d\mu < \infty.$$

But this can't be true for all $n \in \mathbb{N}$ unless $\mu(E) = 0$. □

Proposition 4.2.3. *If $f \colon \mathbb{R} \to [-\infty, \infty]$ is measurable, then f satisfies*

$$\int_{\mathbb{R}} |f| \, d\mu = 0$$

if and only if $f(x) = 0$ almost everywhere.

Proof. Suppose $\int_{\mathbb{R}} |f| \, d\mu = 0$ first. Let $E_n = \{x \in \mathbb{R} : |f(x)| \geq 1/n\}$. Then

$$\frac{1}{n} \chi_{E_n} \leq |f|$$

and so

$$\int_{\mathbb{R}} \frac{1}{n}\chi_{E_n}\,d\mu = \frac{1}{n}\mu(E_n) \leq \int_{\mathbb{R}} |f|\,d\mu = 0.$$

Thus $\mu(E_n) = 0$ for each n. But $E_1 \subseteq E_2 \subseteq \cdots$ and

$$\bigcup_{n=1}^{\infty} E_n = \{x \in \mathbb{R} : f(x) \neq 0\}.$$

So

$$\mu(\{x \in \mathbb{R} : f(x) \neq 0\}) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = 0.$$

Conversely, suppose now that $\mu(\{x \in \mathbb{R} : f(x) \neq 0\}) = 0$. We know $|f|$ is a non-negative measurable function and so there is a monotone increasing sequence $(f_n)_{n=1}^{\infty}$ of measurable simple functions that converges pointwise to $|f|$. From $0 \leq f_n(x) \leq |f(x)|$ we can see that $\{x \in \mathbb{R} : f_n(x) \neq 0\} \subseteq \{x \in \mathbb{R} : f(x) \neq 0\}$ and so $\mu(\{x \in \mathbb{R} : f_n(x) \neq 0\}) \leq \mu(\{x \in \mathbb{R} : f(x) \neq 0\}) = 0$. Being a simple function f_n has a largest value y_n (which is finite) and so if we put $E_n = \{x \in \mathbb{R} : f_n(x) \neq 0\}$ we have

$$f_n \leq y_n\chi_{E_n} \Rightarrow \int_{\mathbb{R}} f_n\,d\mu \leq \int_{\mathbb{R}} y_n\chi_{E_n}\,d\mu = y_n \int_{\mathbb{R}} \chi_{E_n}\,d\mu = y_n\mu(E_n) = 0.$$

From the Monotone Convergence Theorem

$$\int_{\mathbb{R}} |f|\,d\mu = \int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right)\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu = \lim_{n\to\infty} 0 = 0.$$

□

The above result is one way of saying that integration 'ignores' what happens to the integrand on any chosen set of measure 0. Here is a result that says that in way that is often used.

Proposition 4.2.4. *Let $f\colon \mathbb{R} \to [-\infty, \infty]$ be an integrable function and $g\colon \mathbb{R} \to [-\infty, \infty]$ a Lebesgue measurable function with $f(x) = g(x)$ almost everywhere. Then g must also be integrable and $\int_{\mathbb{R}} g\,d\mu = \int_{\mathbb{R}} f\,d\mu$.*

Proof. Let $E = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ (think of E as standing for 'exceptional') and note that $f(x) = g(x)$ almost everywhere means $\mu(E) = 0$.

Write $f = (1 - \chi_E)f + \chi_E f$. Note that both $(1 - \chi_E)f$ and $\chi_E f$ are integrable because they are measurable and satisfy $|(1 - \chi_E)f| \leq |f|$ and $|\chi_E f| \leq |f|$. Also

$$\left|\int_{\mathbb{R}} \chi_E f\,d\mu\right| \leq \int_{\mathbb{R}} |\chi_E f|\,d\mu = 0$$

as $\chi_E f = 0$ almost everywhere. Similarly $\int_{\mathbb{R}} \chi_E g\,d\mu = 0$.

So

$$\begin{aligned}\int_{\mathbb{R}} f\,d\mu &= \int_{\mathbb{R}} ((1-\chi_E)f + \chi_E f)\,d\mu\\ &= \int_{\mathbb{R}} (1-\chi_E)f\,d\mu + \int_{\mathbb{R}} \chi_E f\,d\mu\\ &= \int_{\mathbb{R}} (1-\chi_E)f\,d\mu + 0\\ &= \int_{\mathbb{R}} (1-\chi_E)g\,d\mu\end{aligned}$$

The same calculation (with $|f|$ in place of f) shows $\int_{\mathbb{R}}(1-\chi_E)|g|\,d\mu = \int_{\mathbb{R}}|f|\,d\mu < \infty$, so that $(1-\chi_E)g$ must be integrable. Thus $g = (1-\chi_E)g + \chi_E g$ is also integrable (because $\int_{\mathbb{R}}|\chi_E g|\,d\mu = 0$ and so $\chi_E g$ is integrable and g is then the sum of two integrable functions). Thus we have $\int_{\mathbb{R}} g\,d\mu = \int_{\mathbb{R}}(1-\chi_E)g\,d\mu + \int_{\mathbb{R}}\chi_E g\,d\mu = \int_{\mathbb{R}} f\,d\mu + 0$. □

Remark 4.2.5. It follows that we should be able to manage without allowing integrable functions to have the values $\pm\infty$. The idea is that, if f is integrable, it must be almost everywhere finite. If we change all the values where $|f(x)| = \infty$ to 0 (say) we are only changing f on a set of x's of measure zero. This is exactly changing f to the $(1-\chi_E)f$ in the above proof. The changed function will be almost everywhere the same as the original f, but have finite values everywhere. So from the point of view of the integral of f, this change is not significant.

However, it can be awkward to have to do this all the time, and it is better to allow $f(x) = \pm\infty$.

4.3 Dominated convergence theorem

Theorem 4.3.1 (Lebesgue dominated convergence theorem)**.** *Suppose $f_n\colon \mathbb{R} \to [-\infty,\infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrable $g\colon \mathbb{R} \to [0,\infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n\,d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n\,d\mu = \int_{\mathbb{R}} f\,d\mu$$

Proof. Since $|f_n(x)| \leq g(x)$ and g is integrable, $\int_{\mathbb{R}}|f_n|\,d\mu \leq \int_{\mathbb{R}} g\,d\mu < \infty$. So f_n is integrable. We know f is measurable (as a pointwise limit of measurable functions) and then, similarly, $|f(x)| = \lim_{n\to\infty}|f_n(x)| \leq g(x)$ implies that f is integrable too.

The proof does not work properly if $g(x) = \infty$ for some x. We know that $g(x) < \infty$ almost everywhere. So we can take $E = \{x \in \mathbb{R} : g(x) = \infty\}$ and multiply g and each of the functions f_n and f by $1-\chi_E$ to make sure all the functions have finite values. As we are changing them all only on the set E of measure 0, this change does not affect the integrals or the conclusions. We assume then all have finite values.

Let $h_n = g - f_n$, so that $h_n \geq 0$. By Fatou's lemma

$$\liminf_{n\to\infty} \int_{\mathbb{R}} (g - f_n)\, d\mu \geq \int_{\mathbb{R}} \liminf_{n\to\infty} (g - f_n)\, d\mu = \int_{\mathbb{R}} (g - f)\, d\mu$$

and that gives

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}} g\, d\mu - \int_{\mathbb{R}} f_n\, d\mu \right) = \int_{\mathbb{R}} g\, d\mu - \limsup_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \geq \int_{\mathbb{R}} g\, d\mu - \int_{\mathbb{R}} f\, d\mu$$

or

$$\limsup_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \leq \int_{\mathbb{R}} f\, d\mu \tag{1}$$

Repeat this Fatou's lemma argument with $g + f_n$ rather than $g - f_n$. We get

$$\liminf_{n\to\infty} \int_{\mathbb{R}} (g + f_n)\, d\mu \geq \int_{\mathbb{R}} \liminf_{n\to\infty} (g + f_n)\, d\mu = \int_{\mathbb{R}} (g + f)\, d\mu$$

and that gives

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}} g\, d\mu + \int_{\mathbb{R}} f_n\, d\mu \right) = \int_{\mathbb{R}} g\, d\mu + \liminf_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \geq \int_{\mathbb{R}} g\, d\mu + \int_{\mathbb{R}} f\, d\mu$$

or

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \geq \int_{\mathbb{R}} f\, d\mu \tag{2}$$

Combining (1) and (2) we get

$$\int_{\mathbb{R}} f\, d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \leq \limsup_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu \leq \int_{\mathbb{R}} f\, d\mu$$

which forces

$$\int_{\mathbb{R}} f\, d\mu = \liminf_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu = \limsup_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu$$

and that gives the result because if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ (for a sequence $(a_n)_{n=1}^{\infty}$), it implies that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. $\square$

Remark 4.3.2. The example following Fatou's lemma also shows that the assumption about the existence of the dominating function g can't be dispensed with.

4.4 Applications of the dominated convergence theorem

Theorem 4.4.1 (Continuity of integrals)**.** *Assume $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that $x \mapsto f^{[t]}(x) = f(x,t)$ is measurable for each $t \in \mathbb{R}$ and $t \mapsto f(x,t)$ is continuous for each $x \in \mathbb{R}$. Assume also that there is an integrable $g\colon \mathbb{R} \to \mathbb{R}$ with $|f(x,t)| \leq g(x)$ for each $x, t \in \mathbb{R}$. Then the function $f^{[t]}$ is integrable for each t and the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$F(t) = \int_{\mathbb{R}} f^{[t]}\, d\mu = \int_{\mathbb{R}} f(x,t)\, d\mu(x)$$

is continuous.

Proof. Since $f^{[t]}$ is measurable and $|f^{[t]}| \leq g$ we have $\int_{\mathbb{R}} |f^{[t]}|\,d\mu \leq \int_{\mathbb{R}} g\,d\mu < \infty$ and so $f^{[t]}$ is integrable (for each $t \in \mathbb{R}$). This $F(t)$ makes sense.

To show that F is continuous at $t_0 \in \mathbb{R}$ it is enough to show that for each sequence $(t_n)_{n=1}^\infty$ with $\lim_{n\to\infty} t_n = t_0$ we have $\lim_{n\to\infty} F(t_n) = F(t_0)$.

But that follows from the dominated convergence theorem applied to $f_n(t) = f(x, t_n)$, since we have

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} f(x, t_n) = f(x, t_0)$$

by continuity of $t \mapsto f(x,t)$. We also have $|f_n(t)| = |f(x,t_n)| \leq g(x)$ for each n and each $x \in \mathbb{R}$. $\square$

Example 4.4.2. Show that

$$F(t) = \int_{[0,\infty)} e^{-x}\cos(\pi t)\,d\mu(x)$$

is continuous.

Proof. The idea is to apply the theorem with dominating function $g(x)$ given by

$$g(x) = \chi_{[0,\infty)}(x)e^{-x} = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

We need to know that $\int_{\mathbb{R}} g\,d\mu < \infty$ (and that g is measurable and that $x \mapsto \chi_{[0,\infty)}(x)e^{-x}\cos(\pi t)$ is measurable for each t — but we do know that these are measurable because e^{-x} is continuous and $\chi_{[0,\infty)}$ is measurable).

By the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} g\,d\mu = \lim_{n\to\infty}\int_{\mathbb{R}} \chi_{[-n,n]} g\,d\mu = \lim_{n\to\infty}\int_{\mathbb{R}} \chi_{[0,n]} e^{-x}\,d\mu(x) = \lim_{n\to\infty}\int_0^n e^{-x}\,d\mu(x)$$

You can work this out easily using ordinary Riemann integral ideas and the limit is 1. So $\int_{\mathbb{R}} g\,d\mu < \infty$.

Now the theorem applies because

$$|\chi_{[0,\infty)}(x)e^{-x}\cos(\pi t)| \leq g(x)$$

for each $(x,t) \in \mathbb{R}^2$ (and certainly $t \mapsto \chi_{[0,\infty)}(x)e^{-x}\cos(\pi t)$ is continuous for each x). $\square$

Theorem 4.4.3 (Differentiating under the integral sign)**.** *Assume $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is such that $x \mapsto f^{[t]}(x) = f(x,t)$ is measurable for each $t \in \mathbb{R}$, that $f^{[t_0]}(x) = f(x,t_0)$ is integrable for some $t_0 \in \mathbb{R}$ and $\frac{\partial f(x,t)}{\partial t}$ exists for each (x,t). Assume also that there is an integrable $g\colon \mathbb{R} \to \mathbb{R}$ with $|\frac{\partial f}{\partial t}|_{(x,t)}| \leq g(x)$ for each $x, t \in \mathbb{R}$. Then the function $x \mapsto f(x,t)$ is integrable for each t and the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$F(t) = \int_{\mathbb{R}} f^{[t]}\,d\mu = \int_{\mathbb{R}} f(x,t)\,d\mu(x)$$

is differentiable with derivative

$$F'(t) = \frac{d}{dt}\int_{\mathbb{R}} f(x,t)\,d\mu(x) = \int_{\mathbb{R}} \frac{\partial}{\partial t} f(x,t)\,d\mu(x).$$

Proof. Applying the Mean Value theorem to the function $t \mapsto f(x,t)$, for each $t \neq t_0$ we have to have some c between t_0 and t so that

$$f(x,t) - f(x,t_0) = \frac{\partial f}{\partial t}|_{(x,c)}(t - t_0)$$

It follows that

$$|f(x,t) - f(x,t_0)| \leq g(x)|t - t_0|$$

and so

$$|f(x,t)| \leq |f(x,t_0)| + g(x)|t - t_0|.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}} |f(x,t)|\, d\mu(x) &\leq \int_{\mathbb{R}} (|f(x,t_0)| + g(x)|t - t_0|)\, d\mu(x) \\ &= \int_{\mathbb{R}} |f(x,t_0)|\, d\mu(x) + |t - t_0| \int_{\mathbb{R}} g\, d\mu < \infty, \end{aligned}$$

which establishes that the function $x \mapsto f(x,t)$ is integrable for each t.

To prove the formula for $F'(t)$ consider any sequence $(t_n)_{n=1}^{\infty}$ so that $\lim_{n\to\infty} t_n = t$ but $t_n \neq t$ for each t. We claim that

$$\lim_{n\to\infty} \frac{F(t_n) - F(t)}{t_n - t} = \int_{\mathbb{R}} \frac{\partial}{\partial t} f(x,t)\, d\mu(x). \tag{3}$$

We have

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_{\mathbb{R}} \frac{f(x,t_n) - f(x,t)}{t_n - t}\, d\mu(x) = \int_{\mathbb{R}} f_n(x)\, d\mu(x)$$

where

$$f_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t}.$$

Notice that, for each x we know

$$\lim_{n\to\infty} f_n(x) = \frac{\partial f}{\partial t}|_{(x,t)}$$

and so (3) will follow from the dominated convergence theorem once we show that $|f_n(x)| \leq g(x)$ for each x.

That follows from the Mean Value theorem again because there is c between t and t_n (with c depending on x) so that

$$f_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t} = \frac{\partial f}{\partial t}|_{(x,c)}.$$

So $|f_n(x)| \leq g(x)$ for each x. $\square$

4.5 What's missing?

There are quite a few topics that are very useful and that we have not covered at all. Some of the things we have covered are simplified from the way they are often stated and used.

An example in the latter category is that the Monotone Convergence Theorem and the Dominated Convergence Theorem are true if we only assume the hypotheses are valid almost everywhere. The Monotone Convergence Theorem is still true if we assume that the sequence $(f_n)_{n=1}^\infty$ of measurable functions satisfies $f_n \geq 0$ almost everywhere and $f_n \leq f_{n+1}$ almost everywhere (for each n). Then the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ may exist only almost everywhere. Something similar for the Dominated Convergence Theorem.

We stuck to integrals of functions $f(x)$ defined for $x \in \mathbb{R}$ (or for $x \in X \in \mathscr{L}$ — which is more or less the same because we can extend them to be zero on $\mathbb{R} \setminus X$) and we used only Lebesgue measure μ on the Lebesgue σ-algebra $\mathscr{L}$. What we need abstractly is just a measure space (X, Σ, λ) and Σ-measurable integrands $f\colon X \to [-\infty, \infty]$. By definition f is Σ-measurable if

$$f^{-1}([-\infty, a]) = \{x \in X : f(x) \leq a\} \in \Sigma \quad (\forall a \in \mathbb{R}),$$

and it follows from that condition that $f^{-1}(B) \in \Sigma$ for all Borel subsets $B \subseteq \mathbb{R}$. We can then talk about simple functions on X ($f\colon X \to \mathbb{R}$ with finite range $f(X) = \{y_!, y_2, \ldots, y_n\}$), their standard form ($f = \sum_{j=1}^n y_j \chi_{F_j}$ where $F_j = f^{-1}(\{y_j\})$), integrals of non-negative Σ-measurable simple functions ($\int_X f\,d\lambda = \sum_{j=1}^n y_j \lambda(F_j)$), integrals of non-negative measurable $f\colon X \to [0, \infty]$, (generalised) Monotone convergence theorem, λ-integrable Σ-measurable $f\colon X \to [-\infty, \infty]$, (generalised) Dominated Convergence Theorem.

To make this applicable, we would need some more examples of measures, other that just Lebesgue measure $\mu\colon \mathscr{L} \to [0, \infty]$. We did touch on some examples of measures that don't correspond so obviously to 'total length' as μ does. For instance if $f\colon \mathbb{R} \to [0, \infty]$ is a non-negative (Lebesgue) measurable function, there is an associated measure $\lambda_f\colon \mathscr{L} \to [0, \infty]$ given by $\lambda_f(E) = \int_E f\,d\mu$. (We discussed λ_f when f was simple but the fact that λ_f is a measure even when f is not simple follows easily from the Monotone Convergence Theorem.) If we choose f so that $\int_\mathbb{R} f\,d\mu = 1$, then we get a probability measure from λ_f (and f is called a probability density function). The standard normal distribution is the name given to λ_f when $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. That's just one example.

The Radon Nikodym theorem gives a way to recognise measures $\lambda\colon \mathscr{L} \to [0, \infty]$ that are of the form $\lambda = \lambda_f$ for some non-negative (Lebesgue) measurable function f. The key thing is that $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. The full version of the Radon Nikodym theorem applies not just to measures on $(\mathbb{R}, \mathscr{L})$ with respect to μ, but to many more general measure spaces.

And then we could have explained area measure on $\mathbb{R}^2$, volume measure on $\mathbb{R}^3$, n-dimensional volume measure on $\mathbb{R}^n$, Fubini's theorem relating integrals on product spaces to iterated integrals, the change of variables formula for integrals on $\mathbb{R}^n$, and quite a few more topics.

One place where measure theory comes up is in defining so-called Lebesgue spaces, which are Banach spaces defined using (Lebesgue) integration. For example $L^1(\mathbb{R})$ is the space of integrable functions $f\colon \mathbb{R} \to \mathbb{R}$ with norm $\|f\|_1 = \int_\mathbb{R} |f|\,d\mu$. Or to be more precise it is the space of almost everywhere equivalence classes of such functions. (That is so that $\|f\|_1 = 0$ only for the zero element of the space, a property that norms should have.) To make $L^1(\mathbb{R})$ complete we need the Lebesgue integral. Hilbert spaces like $L^2(\mathbb{R})$ come into Fourier analysis, for instance. By definition $L^2(\mathbb{R})$ is the space of almost everywhere equivalence classes of measurable $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $\int_{\mathbb{R}} |f|^2\,d\mu < \infty$ with norm given by $\|f\|_2 = \left(\int_{\mathbb{R}} |f|^2\,d\mu\right)^{1/2}$.

In short then, there is quite a range of things that the Lebesgue theory is used for (probability theory, Fourier analysis, differential equations and partial differential equations, functional analysis, stochastic processes, ...). My aim was to lay the basis for studying these topics later.