Chapter 3. Lebesgue integral and the monotone convergence theorem

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3.1 Starting point — a σ-algebra and a measure

Remark 3.1.1. What we need to define integrals of $\mathbb{R}$-valued functions, apart from a considerable amount of the terminology we have come across so far, is that

- we have $\mathbb{R}$ and the Borel σ-algebra on it; this will come in for the values of the functions we consider;

- on the domain we need a measure, and a measure needs a σ-algebra. We will use $(\mathbb{R}, \mathscr{L}, \mu)$, where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable sets and $\mu \colon \mathscr{L} \to [0, \infty]$ is the measure given by $\mu(F) = m^*(F)$ for $F \in \mathscr{L}$.

In fact, we could equally well have a more general domain X and we would need a σ-algebra Σ of subsets of X together with a measure $\mu \colon \Sigma \to [0, \infty]$. That is we could treat $\int_X f\,d\mu = \int_X f(x)\,d\mu(x)$, integrals of (certain) functions $f \colon X \to \mathbb{R}$ with general domains X, provided we have (X, Σ, μ), but we will mostly stick to $f \colon \mathbb{R} \to \mathbb{R}$ where $(X, \Sigma, \mu) = (\mathbb{R}, \mathscr{L}, \mu = m^*|_{\mathscr{L}})$.

One advantage of sticking to domain $X = \mathbb{R}$ is that we can picture our functions as graphs $y = f(x)$ in $\mathbb{R}^2$. We could still have that advantage if we allowed $X \subseteq \mathbb{R}$, for instance to define $\int_{[a,b]} f\,d\mu$ where $f \colon [a, b] \to \mathbb{R}$. Actually this case will be included rather easily, but another perspective is to generalise to multiple integrals, that is to integrals of functions $f \colon \mathbb{R}^n \to \mathbb{R}$. We are not ready for that because we only did m^* as length outer measure on $\mathbb{R}$. For $\mathbb{R}^2$, we would need to do something similar to what we did with length to define area measure. The starting point would be areas of rectangles, and we would need to replace what we did with the interval algebra $\mathscr{I}$ by stuff about a 'rectangle algebra'. Some of the details at the start would be a bit different but many of the same proofs can be used in $\mathbb{R}^2$, and for volume outer measure in $\mathbb{R}^3$ and more generally for n-dimensional measure in $\mathbb{R}^n$.

However, what we do in this chapter can be done in considerable generality and we set out the general definitions concisely.

Definition 3.1.2. If X is a set then a collection $\Sigma \subseteq \mathcal{P}(X)$ is called a σ-algebra (of subsets of X) if it satisfies:

(i) $\emptyset \in \Sigma$

(ii) $E \in \Sigma \Rightarrow E^c \in \Sigma$

(iii) $E_1, E_2, \ldots \in \Sigma \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \Sigma$.

A pair (X, Σ) of a set X and a σ-algebra of subsets of X is sometimes called a *measurable space*.

Definition 3.1.3. If Σ is a σ-algebra (of subsets of set X) and $\mu \colon \Sigma \to [0, \infty]$ is a function, then we call μ a *measure* on Σ if it satisfies

(a) $\mu(\emptyset) = 0$

(b) μ is countably additive, that is whenever $E_1, E_2, \ldots \in \Sigma$ are disjoint, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$.

The combination (X, Σ, μ) is called a *measure space* and the subsets $E \subset X$ with $E \in \Sigma$ are called *measurable* subsets (with respect to the given Σ). If in addition $\mu(X) = 1$, then μ is called a *probability measure* and the combination (X, Σ, μ) is called a *probability space*. In the context of probability, measurable subsets $E \in \Sigma$ are called *events* and $\mu(E) \in [0, 1]$ the *probability* of the event E.

Examples 3.1.4.

1. Our primary focus will be $(X, \Sigma, \mu) = (\mathbb{R}, \mathscr{L}, \mu)$ with $\mu = m^*|_{\mathscr{L}}$ being Lebesgue length measure on the σ-algebra $\mathscr{L}$ of Lebesgue measurable subsets of $\mathbb{R}$.

2. If $X \in \mathscr{L}$ then we can define a σ-algebra Σ on X and a measure $\lambda \colon \Sigma \to \mathbb{R}$ by

$$\Sigma = \{F \cap X : F \in \mathscr{L}\}$$

and

$$\lambda(F \cap X) = \mu(F \cap X) = m^*(F \cap X).$$

It may be helpful to have a notation $\mathscr{L}_X$ for this Σ (though that is not a standard notation).

Proof. There is nothing difficult in this except perhaps that there is a possibility of confusion between two notions of complement. Perhaps this can be solved if we write $\mathbb{R} \setminus E$ for the complement of $E \subseteq \mathbb{R}$ and $X \setminus E$ for the complement of $E \subseteq X$ as a subset of X. The notation we usually use of E^c for the complement of E is not very adaptable to more than one meaning of complement.

Notice that, since $X \in \mathscr{L}$, we could also say

$$\Sigma = \{E \in \mathscr{L} : E \subseteq X\}$$

(because $E \in \mathscr{L}$ with $E \subseteq X$ implies $E = E \cap X$, and on the other hand $F \in \mathscr{L}$ implies that $E = F \cap X \in \mathscr{L}$ and has $E \subseteq X$). So $\emptyset \in \Sigma$, $E \in \Sigma \Rightarrow X \setminus E = X \cap (\mathbb{R} \setminus E) \in \Sigma$. Moreover $E_1, E_2, \ldots \in \Sigma$ implies $\bigcup_{n=1}^{\infty} E_n \in \Sigma$.

The properties required for λ to be a measure are just true because we know μ is a measure. $\square$

3. If $X = [0, 1]$, then the previous example turns into an example of a probability space $([0, 1], \Sigma, \lambda)$.

4. If (X, Σ, μ_1) and (X, Σ, μ_2) are two measure spaces with the same underlying set and σ-algebra, and if $c_1, c_2 \geq 0$, then we can get a new measure $\mu = c_1\mu_1 + c_2\mu_2$ on Σ by defining
$$\mu(E) = c_1\mu_1(E) + c_2\mu_2(E) \qquad (E \in \Sigma)$$
(This is easy to check and we can leave it as an exercise.)

3.2 Measurable functions

Definition 3.2.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *Lebesgue measurable* (or measurable with respect to the measurable space $(\mathbb{R}, \mathscr{L})$) if for each $a \in \mathbb{R}$
$$\{x \in \mathbb{R} : f(x) \leq a\} = f^{-1}((-\infty, a]) \in \mathscr{L}.$$

Examples 3.2.2. (i) Continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ are always measurable, because $(-\infty, a]$ is a closed subset and so $f^{-1}((-\infty, a])$ is closed, hence in $\mathscr{L}$.

If you prefer to think about open sets, (a, ∞) is open, so $f^{-1}((a, \infty))$ is open, hence in $\mathscr{L}$, hence its (closed) complement $(f^{-1}((a, \infty)))^c = f^{-1}((a, \infty)^c) = f^{-1}((-\infty, a])$ is in $\mathscr{L}$.

(ii) The characteristic function $\chi_{\mathbb{Q}}$ of the rationals is measurable (but not continuous anywhere).

The reason is that the set $\mathbb{Q} \in \mathscr{L}$ (it is a countable union of single point sets, hence a Borel set, hence in $\mathscr{L}$; another way is to use $m^*(\mathbb{Q}) = 0$ to get $\mathbb{Q} \in \mathscr{L}$) and so then is its complement $\mathbb{R} \setminus \mathbb{Q} \in \mathscr{L}$. For $f = \chi_{\mathbb{Q}}$, we can see that
$$f^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \leq a\} = \begin{cases} \emptyset & \text{if } a < 0 \\ \mathbb{R} \setminus \mathbb{Q} & \text{if } 0 \leq a < 1 \\ \mathbb{R} & \text{if } a \geq 1 \end{cases}$$
and in all cases we get a set in $\mathscr{L}$.

(iii) For a subset $E \subseteq \mathbb{R}$, its characteristic function χ_E is a measurable function if and only if $E \in \mathscr{L}$ (which means E is a measurable set). As in the previous case, for $f = \chi_E$, we can see that
$$f^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \leq a\} = \begin{cases} \emptyset & \text{if } a < 0 \\ \mathbb{R} \setminus E & \text{if } 0 \leq a < 1 \\ \mathbb{R} & \text{if } a \geq 1 \end{cases}$$
Certainly $\emptyset, \mathbb{R} \in \mathscr{L}$ and we have $\mathbb{R} \setminus E \in \mathscr{L} \iff E \in \mathscr{L}$.

Remark 3.2.3. Our aim will be to define integrals for all measurable functions, but we will not quite succeed in that.

Proposition 3.2.4. *If $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then*

$$f^{-1}(B) \in \mathscr{L}$$

for each Borel set B.

Proof. Let
$$\Sigma_f = \{E \subseteq \mathbb{R} : f^{-1}(E) \in \mathscr{L}\}.$$
We claim that Σ_f is a σ-algebra.

That is quite easy to verify.

(i) $f^{-1}(\emptyset) = \emptyset \in \mathscr{L} \Rightarrow \emptyset \in \Sigma_f$

(ii) $E \in \Sigma_f \Rightarrow f^{-1}(E) \in \mathscr{L} \Rightarrow f^{-1}(E^c) = (f^{-1}(E))^c \in \mathscr{L} \Rightarrow E^c \in \Sigma_f$

(iii) $E_1, E_2, \ldots \in \Sigma_f \Rightarrow f^{-1}(E_n) \in \mathscr{L}$ for $n = 1, 2, \ldots$. From this we have $\bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathscr{L}$ and so
$$f^{-1}\left(\bigcup_{n=1}^\infty E_n\right) = \bigcup_{n=1}^\infty f^{-1}(E_n) \in \mathscr{L} \Rightarrow \bigcup_{n=1}^\infty E_n \in \Sigma_f$$

As we know f is Lebesgue measurable, we have

$$(-\infty, a] \in \Sigma_f \qquad (\forall a \in \mathbb{R})$$

and that means that Σ_f contains the σ-algebra generated by $\{(-\infty, a] : a \in \mathbb{R}\}$ — which we know to be the Borel σ-algebra by Corollary 2.3.13. □

Proposition 3.2.5. *If $f, g\colon \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable functions and $c \in \mathbb{R}$, then the following are also Lebesgue measurable functions*

$$cf, f^2, f+g, fg, |f|, \max(f,g)$$

The idea here is to combine functions by manipulating their values at a point. So $fg\colon \mathbb{R} \to \mathbb{R}$ is the function with value at $x \in \mathbb{R}$ given by $(fg)(x) = f(x)g(x)$, and similarly for the other functions.

Proof. cf**:** First if $c = 0$, cf is the zero function, which is measurable (very easy to check that directly). Let $h = cf$ (so that $h(x) = cf(x)$). For $c > 0$, then

$$h^{-1}((-\infty, a]) = \{x \in \mathbb{R} : cf(x) \leq a\} = \{x \in \mathbb{R} : f(x) \leq a/c\} \in \mathscr{L}$$

and so $h = cf$ is Lebesgue measurable. For $c < 0$,

$$h^{-1}((-\infty, a]) = \{x \in \mathbb{R} : f(x) \geq a/c\} = f^{-1}([a/c, \infty)) \in \mathscr{L}$$

using Proposition 3.2.4 and the fact that $[a/c, \infty)$ is a Borel set.

f^2**:** Here we say

$$\{x \in \mathbb{R} : f(x)^2 \leq a\} = \begin{cases} \emptyset & \text{if } a < 0 \\ f^{-1}([-\sqrt{a}, \sqrt{a}]) & \text{if } a \geq 0 \end{cases}$$

$f + g$**:** (This is perhaps the most tricky part.) Fix $a \in \mathbb{R}$ and we aim to show that $\{x \in \mathbb{R} : f(x) + g(x) \leq a\} \in \mathscr{L}$. Taking the complement, this would follow from $\{x \in \mathbb{R} : f(x) + g(x) > a\} \in \mathscr{L}$.

For any x where $f(x) + g(x) > a$ then $f(x) > a - g(x)$ and there is a rational $q \in \mathbb{Q}$ so that $f(x) > q > a - g(x)$. Then $g(x) > a - q$. So

$$x \in S_q = f^{-1}((q, \infty)) \cap g^{-1}((a - q, \infty))$$

for some $q \in \mathbb{Q}$. On the other hand if $f(x) > q$ and $g(x) > a - q$, then $f(x) + g(x) > a$, from which we conclude that

$$\{x \in \mathbb{R} : f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}} S_q.$$

By Proposition 3.2.4 $S_q \in \mathscr{L}$ for each q and hence $\{x \in \mathbb{R} : f(x) + g(x) > a\} \in \mathscr{L}$ (because it is countable union of sets $S_q \in \mathscr{L}$).

fg**:** This follows from the previous parts because $fg = ((f + g)^2 - (f - g)^2)/4$.

$|f|$**:** This is easy as $\{x \in \mathbb{R} : |f(x)| \leq a\} = \emptyset$ if $a < 0$ but is $f^{-1}([-a, a])$ if $a \geq 0$.

$\max(f, g)$**:** $\{x \in \mathbb{R} : \max(f(x), g(x)) \leq a\} = f^{-1}((-\infty, a]) \cap g^{-1}((-\infty, a])$.

□

Remark 3.2.6. It is convenient to allow for functions that sometimes have the value ∞ and sometimes $-\infty$. For that we need to introduce the extended real line $[-\infty, \infty]$, which is $\mathbb{R}$ with two extra elements $-\infty$ and $+\infty$ added. This is similar to our earlier use of $[0, \infty]$.

We order $[-\infty, \infty]$ so that $-\infty < x < \infty$ for each $x \in \mathbb{R}$. We introduce arithmetic on $[-\infty, \infty]$ in a similar way to what we did before for $[0, \infty]$ but we do not allow subtraction of ∞ from itself or $-\infty$ from itself. We extend the usual arithmetic operations on $\mathbb{R}$ with the rules

$$\begin{aligned} x + \infty = \infty + x &= \infty \\ x + (-\infty) = (-\infty) + x &= (-\infty) \qquad (\text{for } x \in \mathbb{R}) \\ x\infty = \infty x &= \infty = (-\infty)(-x) = (-x)(-\infty) \\ x(-\infty) = (-\infty)x &= (-\infty) = (-x)(\infty) = (\infty)(-x) \qquad (\text{for } 0 < x \in \mathbb{R}) \\ 0\infty = \infty 0 &= 0 \\ 0(-\infty) = (-\infty)0 &= 0 \end{aligned}$$

From the point of view of the ordering of $[-\infty, \infty]$, there is a strictly monotone increasing bijection $\phi \colon \mathbb{R} \to (-1, 1)$ given by

$$\phi(x) = \frac{x}{1 + |x|}$$

and both ϕ and ϕ^{-1} are continuous (so that ϕ is a homeomorphism). We can extend ϕ to get an order-preserving bijection $\Phi\colon [-\infty,\infty] \to [-1,1]$ by defining $\Phi(-\infty) = -1$, $\Phi(x) = \phi(x)$ for $x \in \mathbb{R}$ and $\Phi(\infty) = 1$. Since we know that nonempty subsets of $[-1,1]$ have both a supremum and an infimum, it follows by applying Φ^{-1} that all nonempty subsets of $[-\infty,\infty]$ also have a supremum (least upper bound) and infimum (greatest lower bound).

We define convergence on $[-\infty,\infty]$ such that Φ becomes a homeomorphism onto $[-1,1]$ (which means for example that $\lim_{n\to\infty} x_n = \ell$ has the standard meaning if all the terms x_n and the limit ℓ are finite, but $\lim_{n\to\infty} x_n = \infty$ means $\lim_{n\to\infty} \Phi(x_n) = \Phi(\infty) = 1$ or that given any $M < \infty$, there is N so that $M < x_n \leq \infty$ holds for all $n > N$). It is also true that monotone increasing sequences $(x_n)_{n=1}^{\infty}$ in $[-\infty,\infty]$ always converge (to $\sup_n x_n$ in fact) and so also do monotone decreasing sequences always converge (to their infimum).

Definition 3.2.7. A function $f\colon \mathbb{R} \to [-\infty,\infty]$ is called *Lebesgue measurable* (or measurable with respect to the measurable space $(\mathbb{R},\mathscr{L})$) if for each $a \in \mathbb{R}$

$$\{x \in \mathbb{R} : -\infty \leq f(x) \leq a\} = f^{-1}([-\infty,a]) \in \mathscr{L}.$$

Note that this definition agrees with the earlier Definition 3.2.1 if $f(x) \in \mathbb{R}$ always (because then $f^{-1}([-\infty,a]) = f^{-1}((-\infty,a]))$.

Proposition 3.2.8. *Let $f\colon \mathbb{R} \to [-\infty,\infty]$ be a function and let $A = \{x \in \mathbb{R} : f(x) = -\infty\} = f^{-1}(\{-\infty\})$, $B = \{x \in \mathbb{R} : f(x) = \infty\} = f^{-1}(\{\infty\})$. Then the following are equivalent*

(a) f is Lebesgue measurable

(b) $A, B \in \mathscr{L}$ and the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} f(x) & \text{if } -\infty < x < \infty \\ 0 & \text{if } x \in A \cup B \end{cases}$$

is Lebesgue measurable.

Proof. (a) $\Rightarrow$ (b): Note that

$$A = \bigcap_{n=1}^{\infty} f^{-1}([-\infty,-n]) \in \mathscr{L}$$

(because σ-algebras are closed under the operation of taking countable intersections) and

$$\mathbb{R} \setminus B = \bigcup_{n=1}^{\infty} f^{-1}([-\infty,n]) \in \mathscr{L} \Rightarrow B \in \mathscr{L}$$

(countable unions and complements stay in $\mathscr{L}$).

To show that g is measurable, note that

$$g^{-1}((-\infty,a]) = \begin{cases} f^{-1}([-\infty,a]) \cup B & \text{if } a \geq 0 \\ f^{-1}([-\infty,a]) \setminus A & \text{if } a < 0 \end{cases}$$

So $g^{-1}((-\infty, a]) \in \mathscr{L}$ for each a.

(b) $\Rightarrow$ (a): Note that

$$f^{-1}([-\infty, a]) = \begin{cases} g^{-1}((-\infty, a]) \setminus B & \text{if } a \geq 0 \\ g^{-1}((-\infty, a]) \cup A & \text{if } a < 0 \end{cases}$$

and so $f^{-1}([-\infty, a]) \in \mathscr{L}$ for each a (using (b)). □

Remark 3.2.9. We can also define measurability of functions defined on Lebesgue measurable subsets $X \subset \mathbb{R}$, as follows. This would include for example the situation where X is an interval of any kind, or any Borel set (which includes open subsets X of $\mathbb{R}$, in particular).

Definition 3.2.10. If $X \in \mathscr{L}$, then a function $f : X \to [-\infty, \infty]$ is called *Lebesgue measurable on* X if for each $a \in \mathbb{R}$

$$\{x \in X : -\infty \leq f(x) \leq a\} = f^{-1}([-\infty, a]) \in \mathscr{L}.$$

Proposition 3.2.11. *Let $X \in \mathscr{L}$ and $f : X \to [-\infty, \infty]$. Then the following are equivalent*

(a) f is Lebesgue measurable on X

(b) the function $g : \mathbb{R} \to [-\infty, \infty]$ given by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x \in \mathbb{R} \setminus X \end{cases}$$

is Lebesgue measurable on $\mathbb{R}$.

Proof. For $a \in \mathbb{R}$ we have

$$\begin{aligned} g^{-1}([-\infty, a]) &= \{x \in \mathbb{R} : -\infty \leq g(x) \leq a\} \\ &= \begin{cases} (\mathbb{R} \setminus X) \cup \{x \in X : -\infty \leq f(x) \leq a\} & \text{if } a \geq 0 \\ \{x \in X : -\infty \leq f(x) \leq a\} & \text{if } a < 0 \end{cases} \end{aligned}$$

So (a) $\Rightarrow$ (b). For the converse use

$$f^{-1}([-\infty, a]) = \begin{cases} \{x \in \mathbb{R} : -\infty \leq g(x) \leq a\} \setminus (\mathbb{R} \setminus X) & \text{if } a \geq 0 \\ \{x \in \mathbb{R} : -\infty \leq g(x) \leq a\} & \text{if } a < 0 \end{cases}$$

□

We can extend Proposition 3.2.5 to functions with values in $[-\infty, \infty]$ except for one difficulty.

Proposition 3.2.12. *If $X \in \mathscr{L}$, $c \in \mathbb{R}$ and $f, g : X \to [-\infty, \infty]$ are Lebesgue measurable functions on X, then the following are also Lebesgue measurable functions (on X)*

$$cf, f^2, fg, |f|, \max(f, g)$$

If $\{x : f(x) = \infty\} \cap \{x : g(x) = -\infty\} = \emptyset = \{x : f(x) = -\infty\} \cap \{x : g(x) = \infty\}$, then $f + g$ is measurable (on X).

The proof (which we leave as an exercise) is to use Propositions 3.2.11, 3.2.8, and 3.2.5 for cf $(c \neq 0)$, f^2, $f + g$, fg and $|f|$. For $\max(f, g)$ a similar proof can be used.

3.3 Limits of measurable functions

The next result shows that taking (pointwise) limits of sequence of measurable functions still give a measurable limit.

Definition 3.3.1. If $f_n\colon \mathbb{R} \to [-\infty, \infty]$ $(n = 1, 2, \ldots)$ is an infinite sequence of functions, then we say that $f\colon \mathbb{R} \to [-\infty, \infty]$ is the *pointwise limit* of the sequence $(f_n)_n$ if we have

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in \mathbb{R}$.

For any sequence $f_n\colon \mathbb{R} \to [-\infty, \infty]$ we can define $\limsup_{n\to\infty} f_n$ as the function with value at x given by

$$\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\sup_{k\geq n} f_k(x) \right)$$

(something that always makes sense because $\sup_{k\geq n} f_k(x)$ decreases as n increases — or at least does not get any bigger as n increases).

Proposition 3.3.2. *Let $f_n\colon \mathbb{R} \to [-\infty, \infty]$ $(n = 1, 2, \ldots)$ be Lebesgue measurable functions. Then*

(a) the function $x \mapsto \sup_{n\geq 1} f_n(x)$ is Lebesgue measurable;

(b) the function $x \mapsto \inf_{n\geq 1} f_n(x)$ is Lebesgue measurable;

(c) $\limsup_{n\to\infty} f_n$ is Lebesgue measurable; and

(d) if the pointwise limit function $f\colon \mathbb{R} \to [-\infty, \infty]$ exists, then f is Lebesgue measurable.

Proof. (a) If $g(x) = \sup_{n\geq 1} f_n(x)$, then $g^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f_n^{-1}([-\infty, a]) \in \mathscr{L}$ (since the σ-algebra $\mathscr{L}$ is closed under taking countable intersections).

(b) If $g(x) = \inf_{n\geq 1} f_n(x)$, then $-g(x) = \sup_{n\geq 1}(-f_n(x))$ is measurable by (a) and so then is $g(x)$ measurable.

(c) We have $\limsup_{n\to\infty} f_n(x) = \inf_{n\geq 1} g_n(x)$ where $g_n(x) = \sup_{k\geq n} f_k(x)$ is a decreasing sequence. But each g_n is measurable by (b) and then $\limsup_{n\to\infty} f_n(x)$ is measurable by (a).

(d) If the pointwise limit $f(x)$ exists, then $f(x) = \limsup_{n\to\infty} f_n(x)$, which is measurable by (c).

$\square$

3.4 Simple functions

Definition 3.4.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called a *simple function* if the range $f(\mathbb{R})$ is a finite set.

Note that we require finite values.

Proposition 3.4.2. *Each simple function $f\colon \mathbb{R} \to \mathbb{R}$ has a representation*

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

as a linear combination of finitely many characteristic functions of disjoint sets $E_1, E_2, \ldots, E_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Moreover we can assume $a_1, a_2, \ldots, a_n$ are distinct values, that the E_j are all nonempty and that $\bigcup_{j=1}^{n} E_j = \mathbb{R}$. With these assumptions the representation is called the standard representation *and is unique apart from the order of the sum.*

A simple function is Lebesgue measurable if and only if the sets E_j in the standard representation are all in $\mathscr{L}$.

Proof. Since $f(\mathbb{R})$ is a finite set (and can't be empty) we can list the elements $f(\mathbb{R}) = \{y_1, y_2, \ldots, y_n\}$ with $n \geq 1$. Take $E_j = f^{-1}(\{y_j\}) = \{x \in \mathbb{R} : f(x) = y_j\}$ and then we get the standard representation $f = \sum_{i=1}^{n} y_i \chi_{E_i}$.

If f is measurable then $E_j \in \mathscr{L}$ for each j (because one point sets $\{a_j\}$ are Borel sets and we can use Proposition 3.2.4).

Conversely, if each E_j is in $\mathscr{L}$, then each χ_{E_j} measurable, and then so is f measurable by Proposition 3.2.5. □

Notation 3.4.3. We will develop integration first for nonnegative measurable functions $f\colon \mathbb{R} \to [0, \infty]$.

To reduce the case of general (Lebesgue measurable) $f\colon \mathbb{R} \to [-\infty, \infty]$ to the positive case we will use the two associated functions $f^+\colon \mathbb{R} \to [0, \infty]$ and $f^-\colon \mathbb{R} \to [0, \infty]$ given by

$$f^+ = \max(f, 0) = \frac{1}{2}(|f| + f), \quad f^- = \max(-f, 0) = \frac{1}{2}(|f| - f),$$

both of which we know to be measurable. Notice that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

We call f^+ the *positive part* of f and f^- the *negative part* (but note that f^- is positive also).

Definition 3.4.4. If $f\colon \mathbb{R} \to [0, \infty)$ is a nonnegative measurable simple function, with standard representation $f = \sum_{j=1}^{n} a_j \chi_{E_j}$, then we define

$$\int_{\mathbb{R}} f \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j)$$

(where $\mu(E_j)$ means the Lebesgue measure of E_j). (Note that the integral makes sense in $[0, \infty]$.)

For $E \in \mathscr{L}$, we define
$$\int_E f\,d\mu = \int_{\mathbb{R}} \chi_E f\,d\mu$$
(noting that $\chi_E f$ is also a nonnegative measurable simple function).

Examples 3.4.5. (i) For $f = \chi_{[0,1]} + 2\chi_{[4,\infty)}$ the possible values are 0, 1 and 2, and the standard representation is
$$f = \chi_{[0,1]} + 2\chi_{[4,\infty)} + 0\chi_{(-\infty,0)\cup(1,4)}$$
so that
$$\int_{\mathbb{R}} f\,d\mu = 1\mu([0,1]) + 2\mu([4,\infty)) + 0\mu((-\infty,0)\cup(1,4)) = 1 + \infty + 0 = \infty$$

(ii) For $f = \chi_{\mathbb{Q}}$, the standard representation is $f = \chi_{\mathbb{Q}} + 0\chi_{\mathbb{R}\setminus\mathbb{Q}}$ and
$$\int_{\mathbb{R}} f\,d\mu = 1\mu(\mathbb{Q}) + 0\mu(\mathbb{R}\setminus\mathbb{Q}) = 0 + 0 = 0$$

(iii) (Exercise) Show that if $E \in \mathscr{L}$, then $\int_{\mathbb{R}} \chi_E\,d\mu = \mu(E)$.

Proposition 3.4.6. *If $f, g\colon \mathbb{R} \to [0,\infty)$ are measurable simple functions and $\alpha \in [0,\infty)$, then αf, $f+g$ are measurable simple functions and*
$$\int_{\mathbb{R}} \alpha f\,d\mu = \alpha\int_{\mathbb{R}} f\,d\mu, \qquad \int_{\mathbb{R}} (f+g)\,d\mu = \int_{\mathbb{R}} f\,d\mu + \int_{\mathbb{R}} g\,d\mu.$$

Proof. We know from Proposition 3.2.5 that αf, $f+g$ are measurable.

If $\alpha = 0$, then $0f$ has standard representation $0 = 0\chi_{\mathbb{R}}$ and integral $\int_{\mathbb{R}} \alpha f\,d\mu = \int_{\mathbb{R}} 0\chi_{\mathbb{R}}\,d\mu = 0\mu(\mathbb{R}) = 0 = 0\int_{\mathbb{R}} f\,d\mu = \alpha\int_{\mathbb{R}} f\,d\mu$. If $\alpha > 0$ and $f(\mathbb{R}) = \{a_1, a_2, \ldots, a_n\}$ let $F_j = f^{-1}(\{a_j\})$ so that f has standard representation $f = \sum_{j=1}^n a_j\chi_{F_j}$, while αf has standard representation $\alpha f = \sum_{j=1}^n \alpha a_j\chi_{F_j}$. So
$$\int_{\mathbb{R}} \alpha f\,d\mu = \sum_{j=1}^n (\alpha a_j)\mu(F_j) = \alpha\sum_{j=1}^n a_j\mu(F_j) = \alpha\int_{\mathbb{R}} f\,d\mu.$$

To cope with $h = f+g$, let $f(\mathbb{R}) = \{a_1, a_2, \ldots, a_n\}$, $F_j = f^{-1}(\{a_j\})$, $g(\mathbb{R}) = \{b_1, b_2, \ldots, b_m\}$, $G_k = g^{-1}(\{b_k\})$. The range of $h = f+g$ is certainly finite because it is contained in $\{a_j + b_k : 1 \le j \le n, 1 \le k \le m\}$. So h is a simple function. Write $h(\mathbb{R}) = \{c_1, c_2, \ldots, c_p\}$ and then for each $1 \le \ell \le p$ we have
$$H_\ell = h^{-1}(c_\ell) = \bigcup_{\{(j,k) : a_j + b_k = c_\ell\}} F_j \cap G_k.$$

The sets making up H_ℓ are a disjoint union and so note that finite additivity of μ gives us
$$\mu(H_\ell) = \sum_{\{(j,k) : a_j + b_k = c_\ell\}} \mu(F_j \cap G_k)$$

We now have standard representations $f = \sum_{j=1}^{n} a_j \chi_{F_j}$, $g = \sum_{k=1}^{m} b_k \chi_{G_k}$, $h = f + g = \sum_{\ell=1}^{p} c_\ell \chi_{H_\ell}$ and

$$\begin{aligned}
\int_{\mathbb{R}} h\,d\mu &= \sum_{\ell=1}^{p} c_\ell \mu(H_\ell) \\
&= \sum_{\ell=1}^{p} c_\ell \sum_{\{(j,k): a_j + b_k = c_\ell\}} \mu(F_j \cap G_k) \\
&= \sum_{\{(j,k): F_j \cap G_k \neq \emptyset\}} (a_j + b_k)\mu(F_j \cap G_k) \\
&= \sum_{1 \leq j \leq n, 1 \leq k \leq m} (a_j + b_k)\mu(F_j \cap G_k) \\
&\qquad \text{(since } \mu(\emptyset) = 0\text{)} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \mu(F_j \cap G_k) + \sum_{k=1}^{m} \sum_{j=1}^{n} b_k \mu(F_j \cap G_k) \\
&= \sum_{j=1}^{n} a_j \mu\left(\bigcup_{k=1}^{m} F_j \cap G_k\right) + \sum_{k=1}^{m} b_k \mu\left(\bigcup_{j=1}^{n} F_j \cap G_k\right) \\
&\qquad \text{(using finite additivity of } \mu\text{)} \\
&= \sum_{j=1}^{n} a_j \mu(F_j) + \sum_{k=1}^{m} b_k \mu(G_k) \\
&= \int_{\mathbb{R}} f\,d\mu + \int_{\mathbb{R}} g\,d\mu
\end{aligned}$$

□

Remark 3.4.7. The tricky part in the above proof is to stick to the definition of the integral of a simple function in terms of the standard representation. We do this so that the definition is not ambiguous, but now that we have proved this result we can say that if we have any representation of a simple function $f = \sum_{i=1}^{n} y_j \chi_{E_j}$ with $y_j \geq 0$ and E_j measurable, then

$$\int_{\mathbb{R}} f\,d\mu = \sum_{i=1}^{n} y_j \int_{\mathbb{R}} \chi_{E_j}\,d\mu = \sum_{i=1}^{n} y_j \mu(E_j)$$

Corollary 3.4.8. *If $f, g \colon \mathbb{R} \to [0, \infty)$ are measurable simple functions with $f(x) \leq g(x)$ for each x, then*

$$\int_{\mathbb{R}} f\,d\mu \leq \int_{\mathbb{R}} g\,d\mu$$

Proof. $h = g - f$ is measurable (by Proposition 3.2.5), simple (can only have finitely many values) and also $h(x) \geq 0$. So $\int_{\mathbb{R}} h\,d\mu$ is defined and by Proposition 3.4.6,

$$\int_{\mathbb{R}} g\,d\mu = \int_{\mathbb{R}} (f + h)\,d\mu = \int_{\mathbb{R}} f\,d\mu + \int_{\mathbb{R}} h\,d\mu \geq \int_{\mathbb{R}} f\,d\mu.$$

□

Proposition 3.4.9. *If $f\colon \mathbb{R} \to [0,\infty)$ is a measurable simple function, then we can define a measure $\lambda_f\colon \mathscr{L} \to [0,\infty]$ by*

$$\lambda_f(E) = \int_E f\,d\mu$$

Proof. We know we can write f in the form $f = \sum_{j=1}^n a_j \chi_{E_j}$ with $a_j \geq 0$ and $E_j \in \mathscr{L}$ $(1 \leq j \leq n)$. So

$$\chi_E f = \sum_{j=1}^n a_j \chi_E \chi_{E_j} = \sum_{j=1}^n a_j \chi_{E \cap E_j}$$

and

$$\lambda_f(E) = \int_E f\,d\mu = \int_{\mathbb{R}} \chi_E f\,d\mu = \sum_{j=1}^n a_j \mu(E \cap E_j)$$

We can check easily that $\mu_{E_j}(E) = \mu(E \cap E_j)$ defines a measure on $\mathscr{L}$ and it follows from Examples 3.1.4 4 that $\lambda_f = \sum_{j=1}^n a_j \mu_{E_j}$ is a measure. □

Proposition 3.4.10. *If $\lambda\colon \mathscr{L} \to [0,\infty]$ is a measure, then it satisfies*

(a) If $E_1, E_2, \ldots \in \mathscr{L}$ with $E_1 \subseteq E_2 \subseteq \cdots$ (that is a 'monotone increasing sequence' of measurable sets) then

$$\lambda\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n\to\infty} \lambda(E_n) = \sup_n \lambda(E_n)$$

(b) If $E_1, E_2, \ldots \in \mathscr{L}$ with $E_1 \supseteq E_2 \supseteq \cdots$ (that is a 'monotone decreasing sequence' of measurable sets) and if $\lambda(E_1) < \infty$ then

$$\lambda\left(\bigcap_{n=1}^\infty E_n\right) = \lim_{n\to\infty} \lambda(E_n) = \inf_n \lambda(E_n)$$

Proof. (a) (It may help to look at a Venn diagram to see what is happening.) If $(E_n)_{n=1}^\infty$ is a monotone increasing sequence in $\mathscr{L}$ then we can take $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_3 = E_3 \setminus E_2$, in general $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$, so that $F_n \in \mathcal{L}$ for each n, $(F_n)_{n=1}^\infty$ a disjoint sequence (because $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ means that $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$ and so $F_n \cap E_i = \emptyset$ for $i < n$, and that gives $F_n \cap F_i = \emptyset$ as $F_i \subset E_i$).

Now

$$\bigcup_{i=1}^n F_i = E_n$$

for each n and

$$\bigcup_{i=1}^\infty F_i = \bigcup_{n=1}^\infty E_n.$$

By countable additivity of λ (and disjointness of the F_i),

$$\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \lambda\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \lambda(F_i) = \sup_n \sum_{i=1}^{n} \lambda(F_i) = \sup_n \lambda\left(\bigcup_{i=1}^{n} F_i\right) = \sup_n \lambda(E_n)$$

(b) Assume now that $(E_n)_{n=1}^{\infty}$ is a monotone decreasing sequence in $\mathscr{L}$. Let $G_n = E_1 \setminus E_n$ so that $E_1 = G_n \cup E_n$, a disjoint union of sets on $\mathcal{L}$, and

$$\lambda(E_1) = \lambda(G_n) + \lambda(E_n).$$

Using $\lambda(E_1) < \infty$, this implies

$$\lambda(E_n) = \lambda(E_1) - \lambda(G_n).$$

Since $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ we have

$$\lambda\left(\bigcup_{n=1}^{\infty} G_n\right) = \lim_{n\to\infty} \lambda(G_n) = \sup_n \lambda(G_n)$$

by (a). Now,

$$E_1 \setminus \bigcap_{n=1}^{\infty} E_n = E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)^c = \bigcup_{n=1}^{\infty} (E_1 \cap E_n^c) = \bigcup_{n=1}^{\infty} (E_1 \setminus E_n) = \bigcup_{n=1}^{\infty} G_n$$

and, as above,

$$\begin{aligned}
\lambda(E_1) &= \lambda\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) + \lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lambda\left(\bigcup_{n=1}^{\infty} G_n\right) + \lambda\left(\bigcap_{n=1}^{\infty} E_n\right) \\
\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) &= \lambda(E_1) - \lambda\left(\bigcup_{n=1}^{\infty} G_n\right) = \lambda(E_1) - \lim_{n\to\infty} \lambda(G_n) \\
&= \lim_{n\to\infty} \lambda(E_1) - \lambda(G_n) = \lim_{n\to\infty} \lambda(E_n)
\end{aligned}$$

Since $\lambda(E_1) \geq \lambda(E_2) \geq \cdots$, we have $\lim_{n\to\infty} \lambda(E_n) = \inf_n \lambda(E_n)$. □

3.5 Positive measurable functions

Proposition 3.5.1. *If $f\colon \mathbb{R} \to [0, \infty]$ is a (nonnegative extended real valued) measurable function, then there is a monotone increasing sequence $(f_n)_{n=1}^{\infty}$ of (nonnegative) measurable simple functions $f_n\colon \mathbb{R} \to [0, \infty)$ with*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \textit{(for each } x \in \mathbb{R}\textit{)}$$

Proof. For each n, define

$$E_{n,j} = f^{-1}\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)\right) \qquad 1 \le j \le n2^n$$

and

$$E_{n,1+n2^n} = f^{-1}([n,\infty]).$$

Note that each $E_{n,j}$ is in $\mathscr{L}$. Then put

$$f_n = \sum_{j=1}^{n2^n+1} \frac{j-1}{2^n}\chi_{E_{n,j}}.$$

Observe that $f_n(x) \le f(x)$ always. For a given x where $f(x) < \infty$ we have $f_n(x) = n$ for $n \le f(x)$ but for $n > f(x)$ we find that

$$f_n(x) = \frac{1}{2^n}\lfloor 2^n f(x)\rfloor$$

where $\lfloor y\rfloor$ means the greatest integer $\le y$. Since $\lfloor 2^{n+1}f(x)\rfloor \ge 2\lfloor 2^n f(x)\rfloor$, it follows that $f_{n+1}(x) \ge f_n(x)$ always.

Also as $\lfloor 2^n f(x)\rfloor \ge 2^n f(x) - 1$ we get $f_n(x) > f(x) - \frac{1}{2^n}$ as soon as $n > f(x)$. Since $f_n(x) \le f(x)$ we get $\lim_{n\to\infty} f_n(x) = f(x)$ if $f(x) < \infty$.

Finally, if $f(x) = \infty$, then $f_n(x) = n$ for all n and so $\lim_{n\to\infty} f_n(x) = \infty = f(x)$ in this case also. □

Remark 3.5.2. It is tempting to define $\int_{\mathbb{R}} f\,d\mu$ (for $f\colon \mathbb{R} \to [0,\infty]$ a measurable function) as $\lim_{n\to\infty}\int_{\mathbb{R}} f_n\,d\mu$ (where f_n is exactly as in the proof of the above proposition). But that makes it a little tricky to prove that the integral has the various 'obvious' properties it has. So we use a more complicated definition first and prove (as part of the monotone convergence theorem) that we get the same value for the integral.

Another thing to watch out for is that we don't change the definition we already have when $f\colon \mathbb{R} \to \mathbb{R}$ is a measurable simple function.

Definition 3.5.3. For $f\colon \mathbb{R} \to [0,\infty]$ a Lebesgue measurable function, we define

$$\int_{\mathbb{R}} f\,d\mu = \sup\left\{\int_{\mathbb{R}} s\,d\mu : s\colon \mathbb{R} \to [0,\infty) \text{ a measurable simple function with } s \le f\right\}$$

For $E \in \mathscr{L}$, we define

$$\int_E f\,d\mu = \int_{\mathbb{R}} \chi_E f\,d\mu$$

(noting that $\chi_E f$ is also a nonnegative measurable function).

Remark 3.5.4. By Corollary 3.4.8, when $f\colon \mathbb{R} \to \mathbb{R}$ is itself a simple measurable function, this gives the same value for $\int_{\mathbb{R}} f\,d\mu$ (and $\int_E f\,d\mu$) as in the earlier definition (Definition 3.4.4).

Lemma 3.5.5. *For $f, g\colon \mathbb{R} \to [0,\infty]$ Lebesgue measurable functions, we have*

(a) if $f \le g$ (pointwise) then

$$\int_{\mathbb{R}} f\,d\mu \le \int_{\mathbb{R}} g\,d\mu$$

(b) if $E \subset F$ with $E, F \in \mathscr{L}$, then

$$\int_E f\,d\mu \le \int_F f\,d\mu$$

Proof. (a) if $f \le g$, then any simple function $s\colon \mathbb{R} \to [0,\infty)$ with $s \le f$ also has $s \le g$ and so $\int_{\mathbb{R}} s\,d\mu \le \int_{\mathbb{R}} g\,d\mu$. Taking the supremum over all such s we get the result.

(b) Since $E \subset F$, we have $\chi_E f \le \chi_F f$ and so

$$\int_E f\,d\mu = \int_{\mathbb{R}} \chi_E f\,d\mu \le \int_{\mathbb{R}} \chi_F f\,d\mu = \int_F f\,d\mu$$

by (a).

□

Theorem 3.5.6 (Monotone Convergence Theorem)**.** *If $f_n\colon \mathbb{R} \to [0,\infty]$ is a monotone increasing sequence of (Lebesgue) measurable functions with pointwise limit f, then*

$$\int_{\mathbb{R}} f\,d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu$$

Proof. Notice that $f(x) = \lim_{n\to\infty} f_n(x) \in [0,\infty]$ is guaranteed to exist as the sequence $(f_n(x))_{n=1}^\infty$ is monotone increasing. Moreover f is measurable (Proposition 3.3.2 (d)) and so $\int_{\mathbb{R}} f\,d\mu$ is defined by Definition 3.5.3.

From $f_n \le f_{n+1}$ we have $f_j(x) \le \lim_{n\to\infty} f_n(x) = f(x)$ for each j. Hence $\int_{\mathbb{R}} f_j\,d\mu \le \int_{\mathbb{R}} f\,d\mu$ for each j by Lemma 3.5.5. Also $\int_{\mathbb{R}} f_n\,d\mu \le \int_{\mathbb{R}} f_{n+1}\,d\mu$ and so the sequence of integrals $\int_{\mathbb{R}} f_n\,d\mu$ is monotone increasing. That means $\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu$ makes sense in $[0,\infty]$. We have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu = \sup_{n\ge 1} \int_{\mathbb{R}} f_n\,d\mu \le \int_{\mathbb{R}} f\,d\mu.$$

It remains to prove the reverse inequality.

Fix a simple function $s\colon \mathbb{R} \to \mathbb{R}$ with $s(x) \le f(x)$ for all $x \in \mathbb{R}$. We will show $\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,d\mu \ge \int_{\mathbb{R}} s\,d\mu$ and that will be enough because of the way $\int_{\mathbb{R}} f\,d\mu$ is defined.

To show this, fix $\alpha \in (0,1)$ and consider

$$E_{n,\alpha} = \{x \in \mathbb{R} : f_n(x) \ge \alpha s(x)\}$$

Notice that

$$\int_{E_{n,\alpha}} \alpha s\,d\mu \le \int_{E_{n,\alpha}} f_n\,d\mu \le \int_{\mathbb{R}} f_n\,d\mu \tag{1}$$

Because $f_{n+1} \geq f_n$ we know $E_{n,\alpha} \subset E_{n+1,\alpha}$. For $s(x) = 0$, $x \in E_{n,\alpha}$ and for $s(x) > 0$, $\lim_{n\to\infty} f_n(x) = f(x) \geq s(x) > \alpha s(x)$ implies $x \in E_{n,\alpha}$ when n is large enough. So $\bigcup_{n=1}^{\infty} E_{n,\alpha} = \mathbb{R}$ and Proposition 3.4.10 (with Proposition 3.4.9) tells us that

$$\int_{\mathbb{R}} \alpha s \, d\mu = \lambda_{\alpha s}(\mathbb{R}) = \lambda_{\alpha s}\left(\bigcup_{n=1}^{\infty} E_{n,\alpha}\right) = \lim_{n\to\infty} \lambda_{\alpha s}(E_{n,\alpha}) \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$$

(using (1) at the last step). That gives us

$$\alpha \int_{\mathbb{R}} s \, d\mu = \int_{\mathbb{R}} \alpha s \, d\mu \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu.$$

As that is true for each $\alpha \in (0,1)$, it follows that $\int_{\mathbb{R}} s \, d\mu \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$, and so

$$\int_{\mathbb{R}} f \, d\mu = \sup\left\{\int_{\mathbb{R}} s \, d\mu : s \text{ simple measureable, } s \leq f\right\} \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu,$$

completing the proof. □

Corollary 3.5.7. *If $f, g\colon \mathbb{R} \to [0,\infty]$ are (Lebesgue) measurable functions and $\alpha \geq 0$ then*

(a) $\int_{\mathbb{R}} \alpha f \, d\mu = \alpha \int_{\mathbb{R}} f \, d\mu$

(b) $\int_{\mathbb{R}} (f+g) \, d\mu = \int_{\mathbb{R}} f \, d\mu + \int_{\mathbb{R}} g \, d\mu$

Proof. (a) By Proposition 3.5.1, there is a monotone increasing sequence $(f_n)_{n=1}^{\infty}$ of measurable simple functions that converges pointwise to f. By the Monotone Convergence Theorem (3.5.6) we know that $\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$. But also $(\alpha f_n)_{n=1}^{\infty}$ is a sequence of measurable simple functions that converges pointwise to αf and so we also have $\int_{\mathbb{R}} \alpha f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \alpha f_n \, d\mu$. Now from Proposition 3.4.6 we have

$$\alpha \int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \alpha \int_{\mathbb{R}} f_n \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} \alpha f_n \, d\mu = \int_{\mathbb{R}} \alpha f \, d\mu$$

(b) Again by Proposition 3.5.1, there are monotone increasing sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ of measurable simple functions with $\lim_{n\to\infty} f_n(x) = f(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ for each $x \in \mathbb{R}$. By the Monotone Convergence Theorem (3.5.6) we know that $\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu$ and $\int_{\mathbb{R}} g \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} g_n \, d\mu$. But also $(f_n + g_n)_{n=1}^{\infty}$ is a monotone increasing sequence of measurable simple functions that converges pointwise to $f+g$ and so we also have $\int_{\mathbb{R}} (f+g) \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} (f_n + g_n) \, d\mu$. From Proposition 3.4.6 we have

$$\begin{aligned}
\int_{\mathbb{R}} (f+g) \, d\mu &= \lim_{n\to\infty} \int_{\mathbb{R}} (f_n + g_n) \, d\mu \\
&= \lim_{n\to\infty} \left(\int_{\mathbb{R}} f_n \, d\mu + \int_{\mathbb{R}} g_n \, d\mu\right) \\
&= \int_{\mathbb{R}} f \, d\mu + \int_{\mathbb{R}} g \, d\mu
\end{aligned}$$

□

Corollary 3.5.8. *If $f, g\colon \mathbb{R} \to [0,\infty]$ are (Lebesgue) measurable functions, $\alpha \geq 0$ and $E \in \mathscr{L}$, then*

$$\int_E (\alpha f + g)\, d\mu = \alpha \int_E f\, d\mu + \int_E g\, d\mu$$

Proof. Apply Corollary 3.5.7 to $\chi_E(\alpha f + g) = \alpha \chi_E f + \chi_E g$ to get

$$\begin{aligned} \int_E (\alpha f + g)\, d\mu &= \int_{\mathbb{R}} \chi_E(\alpha f + g)\, d\mu \\ &= \int_{\mathbb{R}} (\alpha \chi_E f + \chi_E g)\, d\mu \\ &= \alpha \int_{\mathbb{R}} \chi_E f\, d\mu + \int_{\mathbb{R}} \chi_E g\, d\mu \\ &= \alpha \int_E f\, d\mu + \int_E g\, d\mu \quad \square \end{aligned}$$

3.6 Integrable functions

Definition 3.6.1. If $f\colon \mathbb{R} \to [-\infty,\infty]$ is measurable, then we say that f is *Lebesgue integrable* if

$$\int_{\mathbb{R}} f^+\, d\mu < \infty \text{ and } \int_{\mathbb{R}} f^-\, d\mu < \infty.$$

For integrable f, we define

$$\int_{\mathbb{R}} f\, d\mu = \int_{\mathbb{R}} f^+\, d\mu - \int_{\mathbb{R}} f^-\, d\mu$$

Lemma 3.6.2. *If $f\colon \mathbb{R} \to [-\infty,\infty]$ is measurable, then f is Lebesgue integrable if and only if*

$$\int_{\mathbb{R}} |f|\, d\mu < \infty$$

Proof. If f is Lebesgue integrable, then $|f| = f^+ + f^-$ is (Lebesgue) measurable and

$$\int_{\mathbb{R}} |f|\, d\mu = \int_{\mathbb{R}} f^+\, d\mu + \int_{\mathbb{R}} f^-\, d\mu < \infty.$$

Conversely, if f is Lebesgue integrable and $\int_{\mathbb{R}} |f|\, d\mu < \infty$, then $f^+ \leq |f|$ (and we know f^+ is measurable) and so

$$\int_{\mathbb{R}} f^+\, d\mu \leq \int_{\mathbb{R}} |f|\, d\mu < \infty.$$

Similarly $\int_{\mathbb{R}} f^-\, d\mu < \infty$. $\square$

Exercise 3.6.3. If $f\colon \mathbb{R} \to [-\infty,\infty]$ is (Lebesgue) integrable, show that

$$\left| \int_{\mathbb{R}} f\, d\mu \right| \leq \int_{\mathbb{R}} |f|\, d\mu$$

(an integral version of the triangle inequality).

Lemma 3.6.4. *If $g, h\colon \mathbb{R} \to [0, \infty]$ are (Lebesgue) measurable functions with both $\int_{\mathbb{R}} g\, d\mu < \infty$ and $\int_{\mathbb{R}} h\, d\mu < \infty$, and if*

$$\{x \in \mathbb{R} : g(x) = \infty\} \cap \{x \in \mathbb{R} : h(x) = \infty\} = \emptyset,$$

then $f = g - h$ is integrable and

$$\int_{\mathbb{R}} f\, d\mu = \int_{\mathbb{R}} g\, d\mu - \int_{\mathbb{R}} h\, d\mu.$$

Proof. We know that $f(x) = g(x) - h(x)$ is defined everywhere and f is measurable (by Proposition 3.2.12). Also $|f(x)| \leq g(x) + h(x)$ and so (via Lemma 3.5.5)

$$\int_{\mathbb{R}} |f|\, d\mu \leq \int_{\mathbb{R}} (g + h)\, d\mu = \int_{\mathbb{R}} g\, d\mu + \int_{\mathbb{R}} h\, d\mu < \infty.$$

So f is integrable.

From $f = g - h = f^+ - f^-$ we have $f^- + g = f^+ + h$ and so by Corollary 3.5.7 we get

$$\int_{\mathbb{R}} f^-\, d\mu + \int_{\mathbb{R}} g\, d\mu = \int_{\mathbb{R}} (f^- + g)\, d\mu = \int_{\mathbb{R}} (f^+ + h)\, d\mu = \int_{\mathbb{R}} f^+\, d\mu + \int_{\mathbb{R}} h\, d\mu$$

which gives us

$$\int_{\mathbb{R}} f\, d\mu = \int_{\mathbb{R}} f^+\, d\mu - \int_{\mathbb{R}} f^-\, d\mu = \int_{\mathbb{R}} g\, d\mu - \int_{\mathbb{R}} h\, d\mu. \qquad \square$$

Theorem 3.6.5 (Linearity of the integral)**.** *If $f, g\colon \mathbb{R} \to [-\infty, \infty]$ are (Lebesgue) integrable functions and $\alpha \in \mathbb{R}$ then*

(a) αf is integrable and $\int_{\mathbb{R}} \alpha f\, d\mu = \alpha \int_{\mathbb{R}} f\, d\mu$

(b) if $f(x) + g(x)$ is defined for each $x \in \mathbb{R}$, then $f + g$ is integrable and $\int_{\mathbb{R}} (f + g)\, d\mu = \int_{\mathbb{R}} f\, d\mu + \int_{\mathbb{R}} g\, d\mu$

Proof. (a) If $\alpha = 0$, the result is easy. For $\alpha \neq 0$, αf is measurable (by Proposition 3.2.5) and $\int_{\mathbb{R}} |\alpha f|\, d\mu = |\alpha| \int_{\mathbb{R}} |f|\, d\mu < \infty$ so that αf is integrable. If $\alpha > 0$ then $(\alpha f)^+ = \alpha f^+$, $(\alpha f)^- = \alpha f^-$, which gives (using Corollary 3.5.7 (a))

$$\begin{aligned}
\int_{\mathbb{R}} \alpha f\, d\mu &= \int_{\mathbb{R}} \alpha f^+\, d\mu - \int_{\mathbb{R}} \alpha f^-\, d\mu \\
&= \alpha \int_{\mathbb{R}} f^+\, d\mu - \alpha \int_{\mathbb{R}} f^-\, d\mu \\
&= \alpha \left(\int_{\mathbb{R}} f^+\, d\mu - \int_{\mathbb{R}} f^-\, d\mu \right) \\
&= \alpha \int_{\mathbb{R}} f\, d\mu
\end{aligned}$$

If $\alpha < 0$, then $(\alpha f)^+ = -\alpha f^- = |\alpha| f^-$, $(\alpha f)^- = -\alpha f^+ = |\alpha| f^+$ and so $\alpha f = (\alpha f)^+ - (\alpha f)^- = |\alpha| f^- - |\alpha| f^+$. Using Corollary 3.5.7 (a) we have

$$\begin{aligned} \int_{\mathbb{R}} \alpha f \, d\mu &= \int_{\mathbb{R}} |\alpha| f^- \, d\mu - \int_{\mathbb{R}} |\alpha| f^+ \, d\mu \\ &= |\alpha| \left(\int_{\mathbb{R}} f^- \, d\mu - \int_{\mathbb{R}} f^+ \, d\mu \right) \\ &= \alpha \left(\int_{\mathbb{R}} f^+ \, d\mu - \int_{\mathbb{R}} f^- \, d\mu \right) \\ &= \alpha \int_{\mathbb{R}} f \, d\mu \end{aligned}$$

(b) $f + g$ is measurable (by Proposition 3.2.5). We know $\int_{\mathbb{R}} f^+ \, d\mu < \infty$, $\int_{\mathbb{R}} f^- \, d\mu < \infty$, $\int_{\mathbb{R}} g^+ \, d\mu < \infty$ and $\int_{\mathbb{R}} g^- \, d\mu < \infty$. Also

$$f + g = (f^+ - f^-) + (g^+ - g^-) = (f^+ + g^+) - (f^- + g^-)$$

and from Lemma 3.6.4 (and Corollary 3.5.7 (b) too) we find

$$\begin{aligned} \int_{\mathbb{R}} (f+g) \, d\mu &= \int_{\mathbb{R}} (f^+ + g^+) \, d\mu - \int_{\mathbb{R}} (f^- + g^-) \, d\mu \\ &= \int_{\mathbb{R}} f^+ \, d\mu + \int_{\mathbb{R}} g^+ \, d\mu - \left(\int_{\mathbb{R}} f^- \, d\mu + \int_{\mathbb{R}} g^- \, d\mu \right) \\ &= \int_{\mathbb{R}} f \, d\mu + \int_{\mathbb{R}} g \, d\mu \end{aligned}$$

□

3.7 Functions integrable on subsets

(Recall Definition 3.2.10.)

Definition 3.7.1. If $X \in \mathscr{L}$ and $f \colon X \to [-\infty, \infty]$ is measurable on X, then we say that f is *integrable on* X if the function $F \colon \mathbb{R} \to [-\infty, \infty]$ given by

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x \in \mathbb{R} \setminus X \end{cases}$$

is integrable and in that case we define

$$\int_X f \, d\mu = \int_X F \, d\mu = \int_{\mathbb{R}} \chi_X F \, d\mu$$

Remark 3.7.2. Recall from Proposition 3.2.11 that F must be (Lebesgue) measurable on $\mathbb{R}$.

It is an easy exercise to show that Theorem 3.6.5 remains true for functions integrable on $X \in \mathscr{L}$ and $\int_X$ in place of $\int_{\mathbb{R}}$.

Example 3.7.3. If $X \in \mathscr{L}$ has $\mu(X) < \infty$ and $f\colon X \to \mathbb{R}$ is a constant function $f(x) = c$, then f is integrable and

$$\int_X f\,d\mu = \int_{\mathbb{R}} c\chi_X\,d\mu = c\int_{\mathbb{R}} \chi_X\,d\mu = c\mu(X).$$

Theorem 3.7.4. *If $f\colon [a,b] \to \mathbb{R}$ is continuous on a finite closed interval, then the Lebesgue integral $\int_{[a,b]} f\,d\mu$ exists and has the same value as the Riemann integral $\int_a^b f(x)\,dx$.*

Proof. Since any such f is necessarily bounded, there is $M \geq 0$ so that $|f(x)| \leq M$ for $a \leq x \leq b$. By considering $f(x) + M$ in place of f, the proof can be reduced to the case where $f(x) \geq 0$ for all $x \in [a,b]$. Thus we assume $f(x) \geq 0$.

For this proof we use step functions, rather than simple functions. A step function is a linear combination $\sum_{j=1}^n \lambda_j \chi_{E_j}$ where $\lambda_j \in \mathbb{R}$ and the sets E_j are intervals. (We will only want finite intervals here.)

Consider the lower sums for f with respect to the partitions of $[a,b]$ into 2^n equal subintervals of length $(b-a)/2^n$. For $1 \leq i \leq 2^n$, let

$$L_{n,i} = \inf\left\{ f(x) : a + \frac{b-a}{2^n}(i-1) \leq x \leq a + \frac{b-a}{2^n} i \right\}$$

so that the corresponding lower sum is

$$L_n(f) = \sum_{i=1}^{2^n} L_{n,i} \frac{b-a}{2^n}$$

We can make a step function $s_n(x)$ with integral equal to this sum by letting $s_n(x) = 0$ for $x \notin [a,b]$ and

$$s_n(x) = L_{n,i} \text{ for } a + \frac{b-a}{2^n}(i-1) \leq x < a + \frac{b-a}{2^n} i \qquad (1 \leq i \leq 2^n)$$

(we can make $s_n(b) = L_{n,2^n}$ to be careful). We have then $s_n(x) \leq f(x)$ for $x \in [a,b]$ and

$$\int_{\mathbb{R}} s_n\,d\mu = L_n(f).$$

From the way each partition is a refinement of the previous one, it follows that

$$L_{n+1,2i} \geq L_{n,i} \text{ and } L_{n+1,2i-1} \geq L_{n,i}.$$

Thus $s_n(x) \leq s_{n+1}(x)$ for each $x \in \mathbb{R}$.

From uniform continuity of f we can deduce that, given any $\varepsilon > 0$, for all n sufficiently large

$$\sup_{x \in [a,b]} |f(x) - s_n(x)| = \sup_{x \in [a,b]} f(x) - s_n(x) < \varepsilon$$

and
$$0 < \int_a^b f(x)\,dx - L_n(f) < (b-a)\varepsilon.$$
So we deduce that
$$\lim_{n\to\infty} s_n(x) = F(x) = \begin{cases} f(x) & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b] \end{cases}$$
and
$$\lim_{n\to\infty} L_n(f) = \int_a^b f(x)\,dx$$
But by the monotone convergence theorem (remember we assumed $f(x) \geq 0$)
$$\lim_{n\to\infty} L_n(f) = \lim_{n\to\infty} \int_{\mathbb{R}} s_n\,d\mu = \int_{\mathbb{R}} F\,d\mu = \int_{[a,b]} f\,d\mu$$
which shows that the two integrals are the same. □