Chapter 1. Some requisites

The idea here is to study a few things that are not directly part of our topic of integration, but they are things that will come up pretty soon along the way.

Contents

1.1   Countable and uncountable sets

You probably know very well that some sets are finite and others are infinite. The point here is that some infinite sets are more infinite than others. So there are more real numbers than rational numbers, that is the set $\mathbb{R}$ has a 'greater number' of elements than the set $\mathbb{Q}$. On the other hand $\mathbb{Q}$ has the 'same number' of elements as the integers $\mathbb{Z}$, even though there are many rationals that are not integers. So $\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$ but the first two are the 'same size' while $\mathbb{R}$ is 'bigger'. To explain that, we need to approach the matter systematically.

A rather abstract way of saying that a set S has n elements is to say that there is a bijection $f\colon \{1, 2, \ldots, n\} \to S$. Here the idea is that $n \in \mathbb{N}$ is a (finite) natural number. Another, perhaps more down to earth, way to express that is that $S = \{f(1), f(2), \ldots, f(n)\}$ so that we can list the elements of S in a list of n elements, without repetitions $S = \{s_1, s_2, \ldots, s_n\}$.

You should be able to see also that the idea of listing out the elements $S = \{s_1, s_2, \ldots, s_n\}$, which is probably what you would have said it means for S to have n elements, is just the same as having a bijection $f\colon \{1, 2, \ldots, n\} \to S$. (We can define $f(i) = s_i$ for $i = 1, 2, \ldots, n$.)

How much further can we go? In the trivial direction if S is the empty set, we can say it has zero elements. That somehow corresponds to the case $n = 0$ above. What about going for bigger sets?

We say that a set S is *finite* if it has n elements for some $n = 0, 1, 2, \ldots$. We say S is *infinite* if it is **not** finite.

Then we say that two sets S and T have the 'same number' of elements if there is a bijections $f\colon S \to T$. (The idea is that the bijection gives us a way to pair off elements $s \in S$ with elements $f(s) \in T$, and it seems to make sense then that they have equally as many elements.)

Definition 1.1.1. A set S is called *countably infinite* if there is a bijection $f\colon \mathbb{N} \to S$.

A set S is called *countable* if it is **either** finite **or** countably infinite.

Remark 1.1.2. The definition can be summarised by saying a countable set S is one where all the elements can be listed

$$S = \{s_1, s_2, \ldots\}$$

(without repetitions) in a list that is either finite or infinite.

Examples 1.1.3.   (i) The set of integers $\mathbb{Z}$ is a countably infinite set.

Proof. We can list all the integers $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.   □

Every integer, positive or negative occurs in the list and occurs just once. We can use this to define a bijection $f\colon \mathbb{N} \to \mathbb{Z}$ by $f(1) = 0$, $f(2n) = n$, $f(2n + 1) = -n$ (for $n = 1, 2, 3, \ldots$). So we could make a more formal proof by checking that this is a bijection.

(ii) The set $\mathbb{Q}$ of rational numbers is a countably infinite set.

Proof. We can list the rationals in $\mathbb{Z}$, then the rationals with denominator 2 that are not whole numbers, then the rationals with denominator 3 that are not in the previous lists, then those with denominator 4 not listed already, etc. We get

$$
\begin{array}{cccccc}
0 & 1 & -1 & 2 & -2 & 3 & \cdots \\
\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{5}{3} & \cdots \\
\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & \cdots \\
\vdots
\end{array}
$$

We can then combine all these lists into a single list (of all the rationals) by using a zig-zag pattern.

$$
\begin{array}{ccccccc}
0 & \to & 1 & & -1 & \to & 2 & & -2 & \to & 3 & \cdots \\
& \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
\frac{1}{2} & & -\frac{1}{2} & & \frac{3}{2} & & -\frac{3}{2} & & \frac{5}{3} & & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
\frac{1}{3} & & -\frac{1}{3} & & \frac{2}{3} & & -\frac{2}{3} & & \frac{4}{3} & & & \cdots \\
& \swarrow & & \nearrow \\
\frac{1}{4} & & -\frac{1}{4} \\
\downarrow & \nearrow \\
\frac{1}{5} \\
\vdots
\end{array}
$$

That is

$$
\mathbb{Q} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, -\frac{1}{2}, -1, 2, \frac{3}{2}, -\frac{1}{3}, \ldots \right\}
$$

will be a complete list of the elements of $\mathbb{Q}$.   □

(iii) $[0, 1)$ is an uncountable set.

Proof. To prove this we rely on the fact that each $x \in [0, 1)$ has a decimal expansion

$$
x = 0.d_1 d_2 d_3 d_4 \ldots = \sum_{j=1}^{\infty} \frac{d_j}{10^j}
$$

where each digit $d_j \in \{0, 1, 2, \ldots, 9\}$. A refinement we need is that we never have to use expansions that end in repeating nines. (If we do allow repeating nines, then numbers that have finite decimal expansions have another infinite one, like $0.43 = 0.42999\dot{9}$.) Excluding repeating nines, every number in the range $0 \le x < 1$ has a unique decimal expansion. We'll take this as known.

If we assume now that $[0, 1)$ is countable (which we aim to show is not the case) then we can list all the elements
$$[0, 1) = \{x_1, x_2, x_3, \ldots\}.$$
Each x_n has a decimal expansion (without repeating 9's) which we write out as

$$x_n = 0.d_{n,1}d_{n,2}d_{n,3}d_{n,4}\ldots = \sum_{j=1}^{\infty} \frac{d_{n,j}}{10^j}$$

We now show that the list we made cannot be a complete list of the numbers in the set $[0, 1)$. We produce something that is omitted like this. Put

$$d_n = \begin{cases} 1 & \text{if } d_{n,n} \neq 1 \\ 0 & \text{if } d_{n,n} = 1 \end{cases}$$

and then look at

$$x = 0.d_1 d_2 d_3 d_4 \ldots = \sum_{j=1}^{\infty} \frac{d_j}{10^j}$$

This will be at most as large as $0.111\dot{1} = \sum_{j=1}^{\infty} \frac{1}{10^j} = \frac{1}{9} < 1$ and so $x \in [0, 1)$. But it is nowhere in our list. Certainly x has no repeating nines and $x \neq x_n$ because they differ in the nth decimal place ($d_n \neq d_{n,n}$).

So the list is not a complete list. But that is true no matter what list we start with, showing that $[0, 1)$ is not countable. $\qquad \square$

Proposition 1.1.4. *Subsets of countable sets are countable.*

Proof. Let S be a countable set and let $T \subseteq S$. It is pretty clear that subsets of finite sets are finite and so we can concentrate on the case where S is countably infinite. So the elements of S can be listed in an infinite list
$$S = \{s_1, s_2, s_3, \ldots\}$$
with no repetitions in the list. To get a list of the elements of the subset T, just remove from the list $s_1, s_2, s_3, \ldots$ any elements not in T.

This could be a finite list if T is finite, but still a complete list. $\qquad \square$

Proposition 1.1.5. *The union of two countable sets is countable.*

Proof. Let A and B be countable sets and list their elements in finite or infinite lists

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}.$$

Assuming first that both lists are infinite we can make a list of all the elements of $A \cup B$ by taking

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$$

but that could have repetitions. To eliminate repetitions, skip or remove any term in the list that has already occurred.

   If either A or B is finite, we could put a finite list at the start, then continue with the other, and then remove any repetitions. Of course, if both A and B are finite we will end up with a finite list of the elements of $A \cup B$. $\qquad \square$

Proposition 1.1.6. *Every infinite set S contains a countably infinite subset.*

Proof. Being infinite $S \neq \emptyset$ and so there is an element $s_1 \in S$. Then $S \setminus \{s_1\}$ can't be empty (as $S \neq \{s_1\}$ since S is not finite) and so we can pick $s_2 \in S \setminus \{s_1\}$. Next pick $s_3 \in S \setminus \{s_1, s_2\}$ and inductively, once we have $s_1, s_2, \ldots, s_n$ we pick $s_{n+1} \in S \setminus \{s_1, s_2, \ldots, s_n\}$. This produces an infinite sequence of elements of S with no repetitions and so a countably infinite subset

$$\{s_1, s_2, \ldots\} \subseteq S.$$

$\qquad \square$

Definition 1.1.7. We say that two sets S and T have the the 'same number of elements' if there is a bijection $f \colon S \to T$.

Proposition 1.1.8. *If S is an infinite set then it has the same number of elements as some proper subset of itself.*

Proof. Pick an infinite sequence $s_1, s_2, \ldots$ of elements of S (making a countably infinite subset) and define $f \colon S \to S \setminus \{s_1\}$ by

$$f(s) = \begin{cases} s & \text{if } s \in S \setminus \{s_1, s_2, \ldots\} \\ s_{n+1} & \text{if } s = s_n \text{ for some } n \end{cases}$$

Then f is a bijection and S has the same number of elements as the subset $S \setminus \{s_1\}$. $\qquad \square$

1.2   Inverse images

We will need this quite a lot. It is not such a complicated idea, but it is worthwhile getting used to it and its properties. We will be using it for $\mathbb{R}$-valued functions mainly, but it works abstractly for functions $f \colon A \to B$ between any two sets.

Definition 1.2.1. If $f\colon A \to B$ is a function and $E \subset B$ is a subset of the co-domain set B, then we define the *inverse image of E under f* to be the set

$$f^{-1}(E) = \{a \in A : f(a) \in E\}$$

Remark 1.2.2. What is perhaps confusing is that the notation $f^{-1}(E)$ makes sense even though the inverse function f^{-1} need not exist at all.

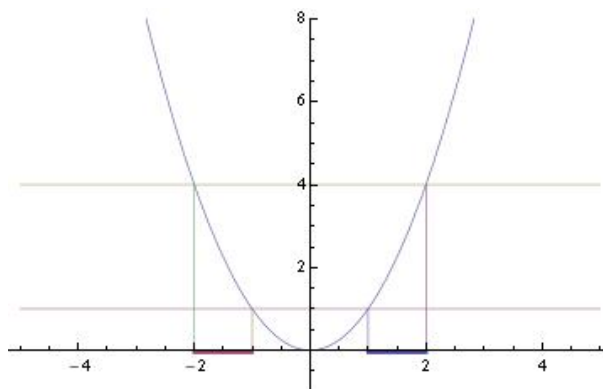Another word that can be used, instead of 'inverse image' is 'pullback'.

Examples 1.2.3. (i) If $f\colon \mathbb{R} \to \mathbb{R}$ is $f(x) = x^2$, then

$$f^{-1}([1,4]) = [1,2] \cup [-2,-1]$$

We can see that from the formula

$$
\begin{aligned}
f^{-1}([1,4]) &= \{x \in \mathbb{R} : f(x) \in [1,4]\} \\
&= \{x \in \mathbb{R} : x^2 \in [1,4]\} \\
&= \{x \in \mathbb{R} : \sqrt{x^2} = |x| \in [1,2]\}
\end{aligned}
$$

or by looking at the graph to see where are the x values with $y = x^2$ between $y = 1$ and $y = 4$:

Exercise. What is $f^{-1}((1,4])$?

Another example (with the same f) is $f^{-1}([-10,-5])$. There are no points x with $f(x) = x^2 < 0$ and so $f^{-1}([-10,-5]) = \emptyset$.

(ii) Let $f\colon \{1,2,5,11\} \to \{8,9,10,11,12\}$ be given by the rule

$$f(1) = 9,\ f(2) = 11,\ f(5) = 9,\ f(11) = 12$$

What is $f^{-1}(\{8,9,10\})$?

We can just go through the elements of $\{1,2,5,11\}$ one by one to see which are included and which not. $f(1) = 9 \in \{8,9,10\}$ and so 1 is included. $f(2) = 11 \notin \{8,9,10\}$ so 11 is not included. Similarly 5 is included, but 11 is not.

$$f^{-1}(\{8,9,10\}) = \{1,5\}.$$

Remark 1.2.4 (Forward images). One can also look at (forward) images of sets under a function $f\colon A \to B$. One such image is the whole range of f, that is

$$f(A) = \{f(a) : a \in A\} = \{b \in B : \exists a \in A \text{ with } f(a) = b\}$$

but we can do something similar for any subset $A_0 \subset A$. We get

$$f(A_0) = \{f(a) : a \in A_0\} = \{b \in B : \exists a \in A_0 \text{ with } f(a) = b\}$$

In the example $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, we have

$$f(\mathbb{R}) = \{y \in \mathbb{R} : y \geq 0\}$$

and also

$$f([0, \infty)) = \{y \in \mathbb{R} : y \geq 0\} = f(\mathbb{R})$$

In this case

$$f(f^{-1}([-4, 4])) = f([0, 2]) = [0, 4] \subsetneq [-4, 4].$$

In general if $f\colon A \to B$ is a function between two sets, and if $E \subseteq B$, then

$$f(f^{-1}(E)) \subseteq E$$

but they need not be equal. They would be equal if f is surjective and in general one could check that $f(f^{-1}(E)) = E \cap f(A)$.

Remark 1.2.5. We will have much more use for inverse images, and many nice things happen for inverse images.

An example is that if $f\colon A \to B$ is a function and $E_1, E_2 \subset B$ are two subsets, then it is always true that

$$f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2) \text{ and } f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2).$$

Can you see that the statement about intersections is not true for forward images? [Hint: Look at $f(x) = x^2$.]
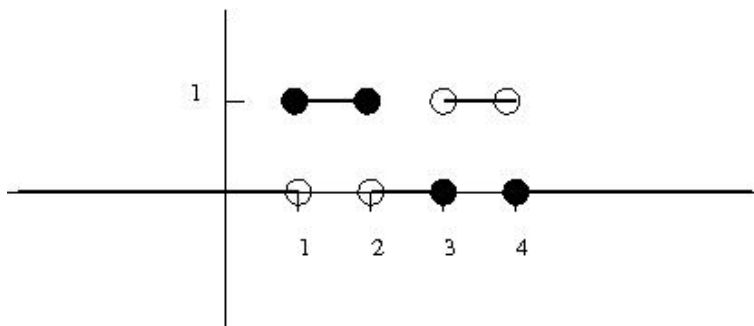
1.3   Characteristic functions

If $E \subseteq S$ is a subset of a set S, then the *characteristic function* of E is the function

$$\begin{aligned} \chi_E \colon S &\to \{0, 1\} \subset \mathbb{R} \\ \chi_E(s) &= \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases} \end{aligned}$$

So, if we take an interval like $E = [1, 2] \subset \mathbb{R}$, then we can graph the characteristic function $\chi_{[1,2]} \colon \mathbb{R} \to \mathbb{R}$. The graph $y = \chi_{[1,2]}(x)$ has $y = 1$ when x is between $x = 1$ and $x = 2$ (inclusive) but $y = 0$ outside that interval.

A slightly more complicated example is $E = [1, 2] \cup (3, 4)$ and for that the graph is

Sometimes people use the term 'indicator function' rather than 'characteristic function'. You should be able to see that if you know the characteristic function χ_E, then you must know the subset E. In fact E is the set of points x in the domain where $\chi_E(x) = 1$. In fancier language, E is the inverse image of $\{1\}$ under χ_E

$$E = (\chi_E)^{-1}(\{1\})$$

Example 1.3.1. The characteristic function of the rationals is a standard example of a bad function from the point of view of the Riemann integral. The graph $y = \chi_{\mathbb{Q}}(x)$ (for $x \in \mathbb{R}$) is not so easy to imagine, and certainly not possible to draw convincingly.

We have to plot $y = 1$ over rational x and $y = 0$ over irrational x. So the graph will be quite a bit of the line $y = 1$ and quite a bit of the line $y = 0$.

Well, you could argue it is more of the line $y = 0$ because the irrationals make an uncountable set and the rationals are a smaller (countably infinite set).

The Riemann integral $\int_0^1 \chi_{\mathbb{Q}}(x)\,dx$ does not exist — all upper sums are 1 and all lower sums are 0. We will see later that the Lebesgue integral will make perfect sense (and will be 0).

1.4 Boolean algebra of sets

The operations of union, intersection, complement, set difference, symmetric difference will come up quite soon.

It is quite helpful to draw Venn diagrams of what these things mean.

We consider subsets of a set S. We can call the subsets A and B, say.

Union

$$A \cup B = \{s : s \in A \text{ or } s \in B\}$$

Intersection

$$A \cap B = \{s : s \in A \text{ and } s \in B\}$$

Complement

$$A^c = \{s \in S : s \notin A\}$$

(Notice that the complement depends in the set S we are working in. If $S = [0,1]$ and $A = [0, 1/2]$, then $A^c = (1/2, 1]$, but if $S = \mathbb{R}$ and we have the same A, then $A^c = (-\infty, 0) \cup (1/2, \infty)$.)

Set difference
$$A \setminus B = \{a \in A : a \notin B\}$$

We can also say that $A \setminus B = A \cap B^c$.

So, for instance, if $A \cap B = \emptyset$, then $A \setminus B = A$.

Another way to think of it is $A \setminus B = A \setminus (A \cap B)$. So to get $A \setminus B$, take away from A any elements of A that are in B (but elements of B that are not in A at all don't get taken away, and don't come into it).

You can describe complements in terms of set difference: $A^c = S \setminus A$.

Symmetric difference
$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

This is the same as $(A \cup B) \setminus (A \cap B)$

Note: Starting with $A \subseteq S$, the complement A^c is another subset of S. Staring with two subsets $A, B \subseteq S$, we get new subsets from $A \cup B$ and $A \cap B$ (and also from $A \setminus B$, $B \setminus A$ and $A \Delta B$, but these are things we can make from complements, unions and intersections). So it is a kind of 'algebra' or manipulation with subsets.

Exercise: These operations can be expressed in terms of characteristic functions. Show for example that

$$\chi_{A \cap B}(s) = \min(\chi_A(s), \chi_B(s)) = \chi_A(s)\chi_B(s)$$
$$\chi_{A \cup B}(s) = \max(\chi_A(s), \chi_B(s)) = \chi_A(s) + \chi_B(s) - \chi_{A \cap B}(s)$$
$$\chi_{A^c}(s) = 1 - \chi_A(s)$$

1.4.1   De Morgan's laws

These say that the complement of a union of subsets is the intersection of the complements, and the complement of an intersection of subsets is the union of the complements. For two subsets $A, B \subseteq S$, they say

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c,$$

but they actually hold for unions and complements of any collection of subsets, even infinite collections of subsets.

If we want to talk about a collection of subsets of a set S, without specifying how many there are, we will usually assume that the subsets are indexed or labelled by elements of some other set I. So we will think of subsets

$$A_i \subseteq S \qquad (\text{for } i \in I)$$

and then their union and intersection are given by these definitions:

$$\bigcup_{i \in I} A_i = \{s : \text{ there is some } i \in I \text{ with } s \in A_i\}$$

$$\bigcap_{i \in I} A_i = \{s : s \in A_i \text{ holds for every } i \in I\}$$

De Morgan's laws apply:

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$