Chapter 0. What is the Lebesgue integral about?

The plan is to have a tutorial sheet each week, most often on Friday, (to be done during the class) where you will try to get used to the ideas introduced in the previous 2 hours of lectures by practising with the concepts. Your work will be graded and count 20% towards the final grade (with the remaining 80% determined by the exam).

See the web site `http://www.maths.tcd.ie/~richardt/MA2224/` for useful information.

## Contents

0.1 What is the best theory for integration?

You've seen the Riemann integral and you know that $\int_a^b f(x)\,dx$ makes sense when $f\colon [a,b] \to \mathbb{R}$ is a continuous function on a finite closed interval in $\mathbb{R}$. There is a theory for what the integral means based on Riemann sums, or on upper and lower sums, and after a certain amount of bother we arrive at a point where we can just use integrals. Then we don't seem to have to worry about the theory any more.

Yet the point of what we are going to do here is to do all this theory again in a different way. Why would we do it again? Well it is not so much that there was anything wrong with the Riemann integral. It gives the right answer and is quite satisfactory for nice (continuous) functions on finite closed intervals. There are then various extensions of the notion of the integral to include various 'improper' integrals, like

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx.$$

There are plenty of functions where the Riemann integral makes sense but the integrand is not continuous, like functions with a finite number of finite jump discontinuities. Say $\int_0^4 f(x)\,dx$ where

$$f(x) = \begin{cases} -1 & x < 1 \\ x^2 & 1 \le x \le 2 \\ 5e^{-x} & 2 < x \end{cases}$$
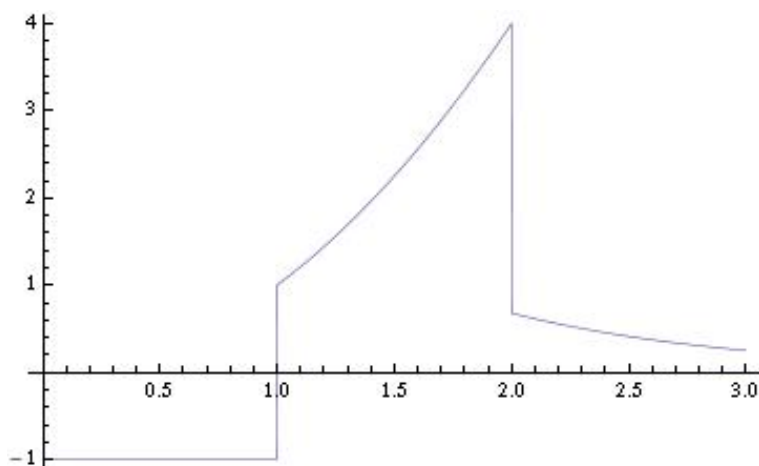
From a practical point of view, there is no real bother with the Riemann integral for a while, but eventually it runs out of steam. For example, in Fourier series things work out very nicely if we work in the Hilbert space of square-integrable functions and we get into bother with that unless we go beyond the Riemann integral. In differential equations and partial differential equations, similar limitations come into play. In probability theory, the Riemann integral is basically inadequate.

There are rivals, but the Lebesgue theory of the integral is almost universally used, and our aim is to try and explain what it is about. We will stick to the one variable case a lot, because we can draw pictures and graphs more easily, and some of the ideas seem more concrete there, but actually most of what we do can be adapted rather easily to very general settings.

0.2   A look back at the Riemann integral

One starts with the picture that the integral $\int_a^b f(x)\,dx$ should be the total area between the graph $y = f(x)$ and the x-axis where $f(x) > 0$ minus the area of the region between the graph and the x-axis where $f(x) < 0$. The bother comes in trying to explain that precisely, and indeed from the fact that we can't allow really bad integrands $f(x)$.

In the Riemann theory we partition the interval $[a, b]$ into (many small) subintervals. We take division points $a_1 < a_2 < \cdots < a_{n-1}$ where $a < a_1$ and $a_{n-1} < b$. To make our life more easy we write $a_0 = a$ and $a_n = b$. We get a picture like this where the vertical lines are at the positions $x = a_i$ ($i = 0, 1, 2, \ldots, n$).
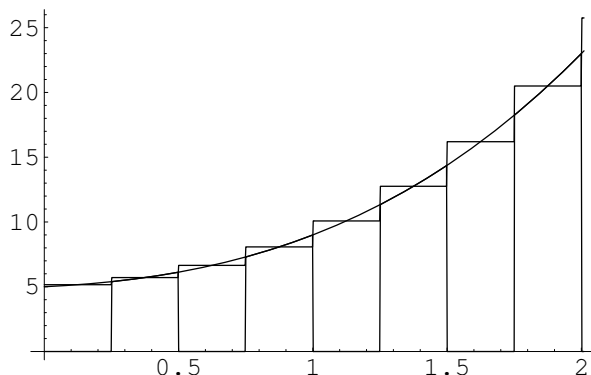
We square off the vertical strips according to one of these strategies:

(i) **Riemann sum.** We pick a value x_i where $a_{i-1} \le x_i \le a_i$ ($i = 1, 2, \ldots, n$) and square off the strips at level $y_i = f(x_i)$. Then the Riemann sum for this setup is

$$\sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) = \sum_{i=1}^{n} y_i(a_i - a_{i-1})$$

(We define the integral then as the limit of these sums as the *mesh size* $\max_{1 \le i \le n}(a_i - a_{i-1})$ goes to zero — provided there is a limit. Using uniform continuity of continuous functions on finite closed intervals, one can show that if the integrand $f(x)$ is continuous on $[a, b]$ and provided the mesh size is small enough, then all the Riemann sums will be close to having the same value, and hence that the limit exists by invoking a Cauchy sequence argument.)

(ii) **Upper sum.** We square off each strip at the highest point on the graph within $a_{i-1} \le x \le a_i$, or to be more precise at the lowest horizontal level $y = y_i$ that is above all the points of the graph in $a_{i-1} \le x \le a_i$. We take

$$y_i = \sup\{f(x) : x \in [a_{i-1}, a_i[\}$$

and then take the upper sum

$$\sum_{i=1}^{n} y_i(a_i - a_{i-1})$$

These upper sums are all too big, or bigger than what we want the integral to be.

(iii) **Lower sum.** We square off each strip at the lowest point on the graph within $a_{i-1} \le x \le a_i$, or to be more precise at the highest horizontal level $y = y_i$ that is below all the points of the graph in $a_{i-1} \le x \le a_i$. We take

$$y_i = \inf\{f(x) : x \in [a_{i-1}, a_i]\}$$

and then take the lower sum

$$\sum_{i=1}^{n} y_i(a_i - a_{i-1})$$

These lower sums are all too small, or smaller than what we want the integral to be.

One can prove that upper sums get smaller if we 'refine' the partition of $[a, b]$, that is if we add in more division points into the list $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$. On the other hand lower sums get bigger when we refine the partitions and it is possible to prove that all lower sums are smaller than all upper sums. We can say that the integral $\int_a^b f(x)\,dx$ exists if we can get the upper sums and lower sums as close together as we like and the integral is the unique number that is trapped between all lower sums and all upper sums.

$$\text{lower sum} \leq \int_a^b f(x)\,dx \leq \text{upper sum}$$

[This approach is followed through in the notes from MA2321 2016–17, `http://www.maths.tcd.ie/~dwilkins/Courses/MA2321/` (Section 3) by Professor Wilkins, where it is also discussed for more general bounded functions on a finite closed interval $[a, b]$, not just continuous functions. One of the results is that $\int_a^b f(x)\,dx$ makes sense for continuous functions $f\colon [a, b] \to \mathbb{R}$. He uses the term *Darboux integral* for this approach to the Riemann integral.]

0.3   Changed approach for the Lebesgue integral

Here is a rough guide to what we will do in this course. There are lots of details to be filled in later.

Instead of partitioning the interval $[a, b]$, the domain over which we are integrating, we instead partition the range. So we draw horizontal lines at (say) spacing $1/N$ apart, maybe at $y = 0, 1/N, 2/N, \ldots$ and also at $y = -1/N, -2/N, \ldots$. As usual there is a trade off between accuracy and the amount of effort, but we should have N very big as we are going to make an approximation, and the approximation will be better if N is big.

We then take the parts of the graph between the levels we have chosen.

Here is a picture for $\int_{\pi/2}^{3\pi} f(x)\,dx = \int_{\pi/2}^{3\pi} \sin x\,dx$.
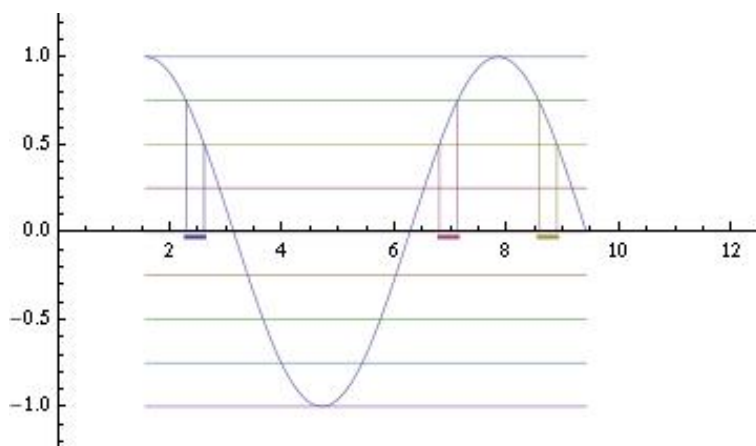
For each horizontal level, say $0.5 \leq y < 0.75$ as in this picture, we replace the graph by a constant and integrate that constant. (We might take the constant value to be 0.5, perhaps. Of course, we have not divided stuff up into narrow enough bands here to get any kind of good result. We should use more and narrower bands.) We might say then that the integral of those parts of the graph should contribute

$$\int_{x \in \{x: 0.5 \leq f(x) < 0.75\}} f(x)\, dx \cong \int_{x \in \{x: 0.5 \leq f(x) < 0.75\}} 0.5\, dx$$

and that should be the total length of $\{x : 0.5 \leq f(x) < 0.75\}$ times the height 0.5.

We add up the result of the same calculation over all the different bands to get an approximate value for $\int_{\pi/2}^{3\pi} f(x)\, dx$ and then we take a limit as the bands get narrower to define the integral.

In this case you may be able to see that the answer we get from the approximation is wrong by at most 0.25 (the heights of the bands we used) times the total width $3\pi - \pi/2$.

Now *what are the obstacles to carrying out this strategy?* One is that the sets we deal with, like $\{x : 0.5 \leq f(x) < 0.75\}$, can get complicated. Then we need to be sure we know what we mean by the lengths of these sets. Well, in the example, we just got 3 little intervals and we are on safe enough ground adding up the 3 lengths, but there are more complicated examples.

If you took $\int_0^1 \sin(1/x)\, dx$ the set $\{x : 0.5 \leq \sin(1/x) < 0.75\}$ would have infinitely many pieces.
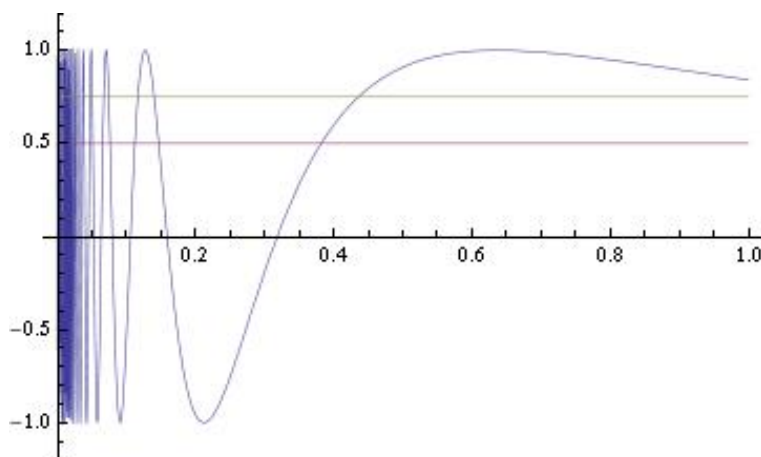
That means we need to be able to add up the infinitely many lengths to get the total length of $\{x \in (0, 1] : 0.5 \leq \sin(1/x) < 0.75\}$ (and of other similar sets). So we will need to be able to find lengths of rather complicated sets and this does end up being rather tricky. In fact this concept of 'measuring' the length of a (possibly) complicated set is probably more than half the difficulty to be sorted out in order to make progress on the Lebesgue theory of the integral.

0.4   The Lebesgue integral

In fact you may not see so clearly that this is the strategy because of the following:

(i) The difficulty mentioned just now of measuring the lengths of complicated sets takes a bit of getting around. There are quite a few details in that. We start with rather easy sets and build up to more complicated ones.

   We also have to allow infinite lengths (say to deal with the length of the whole real line, or an interval that has $\pm\infty$ as one endpoint). And we need to be careful about not allowing sets that are too wild — otherwise things won't work.

(ii) For positive (or never negative) functions we follow more or less the strategy we have outlined above, though it will be expressed in terms of taking limits of sequences of integrals of 'simple' functions.

   Again we can end up with ∞ as the value of the integral of a positive function.

   But for functions that can be negative we adopt a more restrictive approach, not allowing $\pm\infty$ as the integral. What we will do is essentially integrate the function where it is positive separately from where it is negative (using the approach for positive functions) and then combine these values (provided both are finite).

Perhaps it will help to refer back to this little sketch now and again so as to see how far we have progressed.

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