## MA1S12 (Timoney) Tutorial sheet 8b [March 19–24, 2014]

## Name: Solutions

1. Find the equation of the line that is the best least squares fit to the data points (2, -1), (3, 1), (5, 2), (6, 3).

Solution: We take

$$X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} m \\ c \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

and solve the normal equations  $X^t X \mathbf{L} = X^t \mathbf{y}$ .

We need to calculate

$$X^{t}X = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 74 & 16 \\ 16 & 4 \end{bmatrix}$$

and

$$X^{t}\mathbf{y} = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 29 \\ 5 \end{bmatrix}$$

And now solve

$$\begin{bmatrix} 74 & 16\\ 16 & 4 \end{bmatrix} \begin{bmatrix} m\\ c \end{bmatrix} = \begin{bmatrix} 29\\ 5 \end{bmatrix}$$

We can do that by row reducing

$$\begin{bmatrix} 74 & 16 & : 29 \\ 16 & 4 & : 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 8/37 & : 29/74 \\ 16 & 4 & : & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 8/37 & : & 29/74 \\ 0 & 20/37 & : & -47/37 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 8/37 & : & 29/74 \\ 0 & 1 & : & -47/20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & : & 9/10 \\ 0 & 1 & : & -47/20 \end{bmatrix}$$

So m = 9/10 and c = -47/20. The line is

$$y = \frac{9}{10}x - \frac{47}{20}$$

2. Consider the ODE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$$

(a) What are all the solutions (also known as the general solution) for the second order equation?

Solution: In this type of problem we solve the quadratic

$$\lambda^2 - 5\lambda + 4 = 0$$

(the quadratic equation associated with the ODE) and that factors

$$(\lambda - 4)(\lambda - 1) = 0$$

The roots are  $\lambda = 4$  and  $\lambda = 1$  and the general solution is

$$y = C_1 e^{4x} + C_2 e^x$$

(with  $C_1$  and  $C_2$  arbitrary constants).

(b) [Use this part as a homework problem. You may not have time to do it in the hour.] Explain how to transform this single second order equation into a system of two first order differential equations (with y as one of the unknown functions) and how to write the system in the form of a single matrix differential equation

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(Give A and  $y_1$  and  $y_2$ .)

Solution: We take  $y_1 = y$  and  $y_2 = \frac{dy}{dx} = \frac{dy_1}{dx}$  and the equation is equivalent to the system

$$\begin{cases} \frac{dy_1}{dx} = y_2\\ \frac{dy_2}{dx} - 5y_2 + 4y_1 = 0\\ \begin{cases} \frac{dy_1}{dx} = y_2\\ \frac{dy_2}{dx} = -4y_1 + 5y_2 \end{cases}$$

In matrix terms

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

So

or

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ dy/dx \end{bmatrix}$$

Richard M. Timoney