MA1S12 (Timoney) Tutorial sheet 8a
[March 19–24, 2014]

Name: Solution

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1. Find the equation of the line that is the best least squares fit to the data points $(2,3)$, $(3,2)$, $(5,1)$, $(6,-1)$.

   Solution: We take

$$X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} m \\ c \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

   and solve the normal equations $X^t X \mathbf{L} = X^t \mathbf{y}$.

   We need to calculate

$$X^t X = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 74 & 16 \\ 16 & 4 \end{bmatrix}$$

   and

$$X^t \mathbf{y} = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

   And now solve

$$\begin{bmatrix} 74 & 16 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

   We can do that by row reducing

$$\left[\begin{array}{ccc} 74 & 16 & : 11 \\ 16 & 4 & : \ 5 \end{array} \right] \to \left[\begin{array}{ccc} 1 & 8/37 & : 11/74 \\ 16 & 4 & : \quad 5 \end{array} \right] \to \left[\begin{array}{ccc} 1 & 8/37 & : 11/74 \\ 0 & 20/37 & : 97/37 \end{array} \right]$$
$$\to \left[\begin{array}{ccc} 1 & 8/37 & : 11/74 \\ 0 & 1 & : 97/20 \end{array} \right] \to \left[\begin{array}{ccc} 1 & 0 & : -9/10 \\ 0 & 1 & : \ 97/20 \end{array} \right]$$

   So $m = -9/10$ and $c = 97/20$. The line is

$$y = -\frac{9}{10}x + \frac{97}{20}.$$

2. Consider the ODE

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$$

(a) Explain how to transform this single second order equation into a system of two first order differential equations (with y as one of the unknown functions) and how to write the system in the form of a single matrix differential equation

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(Give A and y_1 and y_2.)

Solution: We take $y_1 = y$ and $y_2 = \frac{dy}{dx} == \frac{dy_1}{dx}$ and the equation is equivalent to the system

$$\begin{cases} \frac{dy_1}{dx} &= y_2 \\ \frac{dy_2}{dx} - 7y_2 + 10y_1 &= 0 \end{cases}$$

or

$$\begin{cases} \frac{dy_1}{dx} &= y_2 \\ \frac{dy_2}{dx} &= -10y_1 + 7y_2 \end{cases}$$

In matrix terms

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ dy/dx \end{bmatrix}$$

(b) **[Use this part as a homework problem. You may not have time to do it in the hour.]** What are all the solutions (also known as the general solution) for the second order equation?

Solution: The characteristic equation $\det(A - \lambda I_2) = 0$ for the eigenvaluses of A comes down in this case to

$$\lambda^2 - 7\lambda + 10 = 0$$

(the quadratic equation associated with the ODE) and that factors

$$(\lambda - 5)(\lambda - 2) = 0$$

The roots are $\lambda = 5$ and $\lambda = 2$ and the general solution is

$$y = C_1 e^{5x} + C_2 e^{2x}$$

(with C_1 and C_2 arbitary constants).

Richard M. Timoney