

MA1S12 (Timoney) Tutorial sheet 6c

[March 3–7, 2014]

Name: Solutions

1. Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

Solution: The eigenvalues are the solutions of $\det(A - \lambda I_3) = 0$. We compute

$$A - \lambda I_3 = \begin{bmatrix} 6 - \lambda & 0 & 2 \\ 0 & 2 - \lambda & 0 \\ 2 & 0 & 6 - \lambda \end{bmatrix}$$

and if we expand the determinant along the 2nd row (or 2nd column) we get

$$\det(A - \lambda I_3) = (2 - \lambda) \det \begin{bmatrix} 6 - \lambda & 2 \\ 2 & 6 - \lambda \end{bmatrix} = (2 - \lambda)((6 - \lambda)^2 - 4)$$

The roots satisfy $\lambda = 2$ and $\lambda - 6 = \pm 2$. So they are

2, 8 and 4.

Those are the eigenvalues.

2. For the same A , find unit eigenvectors (*i.e.* length 1 eigenvectors) for each eigenvalue.

Solution: For $\lambda = 2$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 2I_3 : 0]$ or

$$\begin{bmatrix} 4 & 0 & 2 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 2 & 0 & 4 & : & 0 \end{bmatrix}$$

The eigenvector will turn out as

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 8$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 8I_3 : 0]$ or

$$\begin{bmatrix} -2 & 0 & 2 & : & 0 \\ 0 & -6 & 0 & : & 0 \\ 2 & 0 & -2 & : & 0 \end{bmatrix}$$

An eigenvector will turn out as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and we should normalise that to

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

For $\lambda = 4$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 4I_3 : 0]$ or

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right]$$

An eigenvector will turn out as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and we should normalise that to

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

3. For the same A , find an orthogonal matrix P and a diagonal matrix D so that $A = PDP^t$.

Solution:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(Other possible answers multiply the columns of P by ± 1 and you can permute 3, 6 and 4 on the diagonal of D if you apply the same permutation to the columns of P .)

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