MA1S12 (Timoney) Tutorial sheet 6a
[March 3–7, 2014]

Name: Solutions

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1. Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Solution: The eigenvalues are the solutions of $\det(A - \lambda I_3) = 0$. We compute

$$A - \lambda I_3 = \begin{bmatrix} 5-\lambda & 1 & 0 \\ 1 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

and if we expand the determinant along the 3rd row (or 3rd column) we get

$$\det(A - \lambda I_3) = (3-\lambda)\det\begin{bmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} = (3-\lambda)((5-\lambda)^2 - 1)$$

The roots satisfy $\lambda = 3$ and $\lambda - 5 = \pm 1$. So they are

$$3, 6 \text{ and } 4.$$

Those are the eigenvalues.

2. For the same A, find unit eigenvectors (*i.e.* length 1 eigenvectors) for each eigenvalue.

Solution: For $\lambda = 3$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 3I_3 : 0]$ or

$$\begin{bmatrix} 2 & 1 & 0 & : & 0 \\ 1 & 2 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

The eigenvector will turn out as

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 6$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 6I_3 : 0]$ or

$$\begin{bmatrix} -1 & 1 & 0 & : & 0 \\ 1 & -1 & 0 & : & 0 \\ 0 & 0 & -3 & : & 0 \end{bmatrix}$$

An eigenvector will turn out as

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and we should normalise that to

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

For $\lambda = 4$ we should solve the system that gives rise to doing Gaussian elimination on $[A - 4I_3 : 0]$ or

$$\begin{bmatrix} 1 & 1 & 0 & : & 0 \\ 1 & 1 & 0 & : & 0 \\ 0 & 0 & -1 & : & 0 \end{bmatrix}$$

An eigenvector will turn out as

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and we should normalise that to

$$\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

3. For the same A, find an orthogonal matrix P and a diagonal matrix D so that $A = PDP^t$.

Solution:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

(Other possible answers multiply the columns of P by ± 1 and you can permute 3, 6 and 4 on the diagonal of D if you apply the same permutatioin to the columns of P.

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