MA1S12 (Timoney) Tutorial sheet 5c
[February 17–21, 2014]

Name: Solutions

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1. Show matrix
$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix}$$
is a rotation matrix. [Hint: Is it orthogonal? What is its determinant?]

   Solution: Rotation matrices are exactly orthogonal matrices of determinant 1. We can see
$$\begin{aligned} RR^t &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} (-1)^1 & 0 & 0 \\ 0 & \cos^2(\pi/3) + \sin^2(\pi/3) & \cos(\pi/3)\sin(\pi/3) - \sin(\pi/3)\cos(\pi/3) \\ 0 & \sin(\pi/3)\cos(\pi/3) - \cos(\pi/3)\sin(\pi/3) & \sin^2(\pi/3) + \cos^2(\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \end{aligned}$$
So R is orthogonal.

$$\det R = (-1)\det \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{bmatrix} = (-1)(-\cos^2(\pi/3) - \sin^2(\pi/3)) = (-1)(-1) = 1$$

   So R is a rotation.

2. For the same R, find the angle $\cos\theta$ for the angle θ of rotation. [Hint: use the trace.]

   Solution: We know that the angle θ of rotation must satisfy
$$\operatorname{trace}(R) = 1 + 2\cos\theta$$
But
$$\operatorname{trace}(R) = -1 + \cos(\pi/3) - \cos(\pi/3) = -1$$
and so we get $1 + 2\cos\theta = -1$, $2\cos\theta = -2$, $\cos\theta = -1$.

3. For the same R, find a (nonzero) vector parallel to the axis of rotation. [Hint: vector fixed by R. For this it helps to remember $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.]

   Solution: The vector $\mathbf{u}$ we want has to satisfy $R\mathbf{u} = \mathbf{u}$ or $(R - I_3)\mathbf{u} = \mathbf{0}$.

   We have

$$R - I_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \cos(\pi/3) - 1 & \sin(\pi/3) \\ 0 & \sin(\pi/3) & -\cos(\pi/3) - 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -3/2 \end{bmatrix}$$

   and so we should row reduce the augmented matrix

$$\begin{bmatrix} -2 & 0 & 0 & :0 \\ 0 & -1/2 & \sqrt{3}/2 & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{bmatrix}$$

   First divide row 1 by -2

$$\begin{bmatrix} 1 & 0 & 0 & :0 \\ 0 & -1/2 & \sqrt{3}/2 & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{bmatrix}$$

   Now multiply row 2 by -2

$$\begin{bmatrix} 1 & 0 & 0 & :0 \\ 0 & 1 & -\sqrt{3} & :0 \\ 0 & \sqrt{3}/2 & -3/2 & :0 \end{bmatrix}$$

   Subtract $\sqrt{3}/2$ times row 2 from row 3

$$\begin{bmatrix} 1 & 0 & 0 & :0 \\ 0 & 1 & -\sqrt{3} & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

   So the components of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ must satisify $u_1 = 0$ and $u_2 - \sqrt{3}u_3 = 0$. (u_3 is the free variable.) Taking $u_3 = 1$ we get $u_2 = \sqrt{3}$ and $\mathbf{u} = \sqrt{3}\mathbf{j} + \mathbf{k}$.

   (That is not a unit vector. If we wanted a unit vector we should take $(\sqrt{3}\mathbf{j}+\mathbf{k})/\|\sqrt{3}\mathbf{j}+\mathbf{k}\| = (\sqrt{3}\mathbf{j} + \mathbf{k})/\sqrt{4} = (1/2)(\sqrt{3}\mathbf{j} + \mathbf{k})$.)

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