MA1S12 (Timoney) Tutorial sheet 5a

[February 17–21, 2014]

Name: Solutions

For the first two questions, let

$$A = \begin{bmatrix} (1+4\sqrt{3})/9 & (5-\sqrt{3})/9 & (1+\sqrt{3})/9 \\ (-1-\sqrt{3})/9 & (8+5\sqrt{3})/18 & (-11+4\sqrt{3})/18 \\ (-5+\sqrt{3})/9 & (-5+4\sqrt{3})/18 & (8+5\sqrt{3})/18 \end{bmatrix}$$

Then A is in fact an orthogonal matrix and $\det(A) = 1$.

1. Show that the axis of rotation for A is parallel to the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

   Solution: Calculate

$$\begin{aligned} A \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{9}(1+4\sqrt{3}+10-2\sqrt{3}-2-2\sqrt{3}) \\ \frac{1}{18}(-2-2\sqrt{3}+16+10\sqrt{3}+22-8\sqrt{3}) \\ \frac{1}{18}(-10+2\sqrt{3}-10+8\sqrt{3}-16-10\sqrt{3}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{9}(9+0\sqrt{3}) \\ \frac{1}{18}(36+0\sqrt{3}) \\ \frac{1}{18}(-36+0\sqrt{3}) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \end{aligned}$$

2. Find $\cos\alpha$ where α is the angle of rotation for A.

   Solution: We know that $\operatorname{trace}(A) = 1 + 2\cos\alpha$. So we get

$$1 + 2\cos\alpha = (1+4\sqrt{3})/9 + (8+5\sqrt{3})/18 + (8+5\sqrt{3})/18 = 1 + \sqrt{3}$$

   Thus $\cos\alpha = \sqrt{3}/2$.

3. Use the Gram-Schmidt method starting with the 3 vectors $\mathbf{u} = (\mathbf{2}i - \mathbf{j} + \mathbf{k})/\sqrt{6}$, $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{s} = \mathbf{k}$.

   Solution:

   Step 1: If $\mathbf{u}$ is not a unit vector already, replace it by $(1/\|\mathbf{u}\|)\mathbf{u}$ (the unit vector with the same direction).

   $\|u\|^2 = (4+1+1)/6 = 1$. So we don't need this step.

   Step 2: Take

$$\mathbf{v} = \frac{\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}}{\|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}\|}$$

   Here $\mathbf{r}\cdot\mathbf{u} = 2 - 2 + 0 = 0$ and so

$$\mathbf{v} = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\mathbf{r}}{\sqrt{4+1}} = (\mathbf{i} + 2\mathbf{j})/\sqrt{5}$$

Step 3: Take

$$\mathbf{w} = \frac{\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v}}{\|\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v}\|}$$

$\mathbf{s} \cdot \mathbf{u} = 1/\sqrt{6}$
$(\mathbf{s} \cdot \mathbf{v} = 0$
$\mathbf{s} - (\mathbf{s} \cdot \mathbf{u})\mathbf{u} - (\mathbf{s} \cdot \mathbf{v})\mathbf{v} = \mathbf{k} - \frac{1}{6}(\mathbf{2}i - \mathbf{j} + \mathbf{k}) = \frac{1}{6}(-2\mathbf{i} + \mathbf{j} + 5\mathbf{k})$
$\| - 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}\| = \sqrt{4 + 1 + 25} = \sqrt{30}$
$\mathbf{w} = \frac{1}{\sqrt{30}}(-2\mathbf{i} + \mathbf{j} + 5\mathbf{k})$

The result of Gram-Schmidt is then the 3 vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Richard M. Timoney