MA1S12 (Timoney) Tutorial sheet 4
[February 10–14, 2014]

Name: Solutions

In this sheet, we consider 3 orthonormal vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ in $\mathbb{R}^3$. Let

$$P = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

1. If A and B are orthogonal $n \times n$ matrices, show that their product AB is also orthogonal. [Hint: What is the transpose of a product?]

   Solution: The transpose of a product is the product of the transposes taken in the reverse order. So $(AB)^t = B^t A^t$.

   To say A is orthogonal means that $A^t = A^{-1}$ or $AA^t = I_n$. Similarly $BB^t = I_n$. To show AB is orthogonal we compute

   $$(AB)(AB)^t = (AB)B^t A^t = A(BB^t)A^t = AI_nA^t = aA^t = I_n$$

   and this shows $(AB)^t$ is the inverse of AB. So AB has to be orthogonal.

2. Show that the rotation matrix

   $$R = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} P^t$$

   is an orthogonal matrix.

   Solution: We know P and P^t are orthogonal and so we can use the previous question if we check that the middle matrix is orthogonal.

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ 0 & -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3
\end{aligned}$$

So now by the first question, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} P^t$$

is an orthogonal matrix, and multiplying by P on the left we get that R is orthogonal.

3. For the same R, show that $\det(R) = 1$.

   Solution: Since the determinant of a product is the product of the determinants

$$\begin{aligned} \det(R) &= \det(P)\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}\right)\det(P^t) \\ &= \det(P)^2 \det\left(\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\right) \\ &\quad \text{(using } \det(P^t) = \det(P) \text{ and expanding along row 1 or column 1)} \\ &= (\pm 1)^2(\cos^2\alpha + \sin^2\alpha) = 1 \end{aligned}$$

   (since $\det(P) = \pm 1$ because P is orthogonal).

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