MA1S12 (Timoney) Tutorial sheet 4c
[February 10–14, 2014]

Name: Solutions

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1. Let $\mathbf{u} = (2/3)\mathbf{i} - (2/3)\mathbf{j} + (1/3)\mathbf{k}$ and $\mathbf{v} = (1/\sqrt{5})\mathbf{j} + (2/\sqrt{5})\mathbf{k}$

   Show that $\mathbf{u}$ and $\mathbf{v}$ are both unit vectors and that they are orthogonal to one another.

   Solution:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{u} &= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1 \text{ (which implies } \|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = 1) \\ \mathbf{v}\cdot\mathbf{v} &= \frac{1}{5} + \frac{4}{5} = 1 \text{ (which implies } \|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = 1) \\ \mathbf{u}\cdot\mathbf{v} &= 0 - \frac{2}{3\sqrt{5}} + \frac{2}{3\sqrt{5}} = 0 \text{ (which means } \mathbf{u} \perp \mathbf{v}) \end{aligned}$$

2. Find a unit vector $\mathbf{w}$ which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (the vectors in the previous question). [Hint: cross products might help here?]

   Solution: The cross product $\mathbf{u}\times\mathbf{v}$ will be perpendicular to each of $\mathbf{u}$ and $\mathbf{v}$ and will have length

$$\|\mathbf{u}\times\mathbf{v}\|\|\mathbf{u}\|\|\mathbf{v}\|\sin\pi/2 = 1$$

   (using the fact that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/2$ and that they are unit vectors). So we should take $\mathbf{w} = \mathbf{u}\times\mathbf{v}$ and compute that

$$\mathbf{w} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2/3 & -2/3 & 1/3 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = -\frac{\sqrt{5}}{3}\mathbf{i} - \frac{4}{3\sqrt{5}}\mathbf{j} + \frac{2}{3\sqrt{5}}\mathbf{k}$$

3. Using the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ from the previous questions, write down a matrix P so that $P\mathbf{i} = \mathbf{u}$, $P\mathbf{j} = \mathbf{v}$ and $P\mathbf{k} = \mathbf{w}$.

   Solution: P should have $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as columns.

$$P = \begin{bmatrix} 2/3 & 0 & -\sqrt{5}/3 \\ -2/3 & 1/\sqrt{5} & -4/(3\sqrt{5}) \\ 1/3 & 2/\sqrt{5} & 2/(3\sqrt{5}) \end{bmatrix}$$

   Using P, write down a product R of 3 matrices so that R represents the transformation of rotation by $\pi/3$ about the axis $\mathbf{u}$ (in such a way that $\mathbf{v}$ rotates by $\pi/3$ towards $\mathbf{w}$).

Solution:

$$\begin{aligned} R &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} P^t \\ &= \begin{bmatrix} 2/3 & 0 & -\sqrt{5}/3 \\ -2/3 & 1/\sqrt{5} & -4/(3\sqrt{5}) \\ 1/3 & 2/\sqrt{5} & 2/(3\sqrt{5}) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 5/3 & -4/(3\sqrt{5}) & 2/(3\sqrt{5}) \end{bmatrix} \end{aligned}$$

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