**MA1S12 (Timoney) Tutorial sheet 4b**
[February 10–14, 2014]

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**Name:** Solutions

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1. If $A$ and $B$ are orthogonal $n \times n$ matrices, show that their product $AB$ is also orthogonal. [Hint: What is the transpose of a product?]

   *Solution:* The transpose of a product is the product of the transposes taken in the reverse order. So $(AB)^t = B^t A^t$.

   To say $A$ is orthogonal means that $A^t = A^{-1}$ or $AA^t = I_n$. Similarly $BB^t = I_n$. To show $AB$ is orthogonal we compute

   $$(AB)(AB)^t = (AB)B^tA^t = A(BB^t)A^t = AI_nA^t = aA^t = I_n$$

   and this shows $(AB)^t$ is the inverse of $AB$. So $AB$ has to be orthogonal.

2. Let $\mathbf{u} = (4/\sqrt{21})\mathbf{i} - (2/\sqrt{21})\mathbf{j} + (1/\sqrt{21})\mathbf{k}$, $\mathbf{v} = (1/\sqrt{5})\mathbf{j} + (2/\sqrt{5})\mathbf{k}$ and $\mathbf{w} = (5/\sqrt{105})\mathbf{i} + (8/\sqrt{105})\mathbf{j} - (4/\sqrt{105})\mathbf{k}$.

   Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are all unit vectors and that they are pairwise perpendicular.

   *Solution:*

   $$\begin{aligned}
   \mathbf{u}\cdot\mathbf{u} &= \frac{16}{21} + \frac{4}{21} + \frac{1}{21} = 1 \text{ (which implies } \|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = 1)\\
   \mathbf{v}\cdot\mathbf{v} &= \frac{1}{5} + \frac{4}{5} = 1 \text{ (which implies } \|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = 1)\\
   \mathbf{w}\cdot\mathbf{w} &= \frac{25}{105} + \frac{64}{105}\frac{16}{105} = \frac{105}{105} = 1 \text{ (which implies } \|\mathbf{w}\| = \sqrt{\mathbf{w}\cdot\mathbf{w}} = 1)\\
   \mathbf{u}\cdot\mathbf{v} &= 0 - \frac{2}{\sqrt{21}\sqrt{5}} + \frac{2}{\sqrt{21}\sqrt{5}} = 0 \text{ (which means } \mathbf{u}\perp\mathbf{v})\\
   \mathbf{u}\cdot\mathbf{w} &= \frac{20}{\sqrt{21}\sqrt{105}} - \frac{16}{\sqrt{21}\sqrt{105}} - \frac{4}{\sqrt{21}\sqrt{105}} = 0 \text{ (which means } \mathbf{u}\perp\mathbf{w})\\
   \mathbf{v}\cdot\mathbf{u} &= \frac{8}{\sqrt{5}\sqrt{105}} - \frac{8}{\sqrt{5}\sqrt{105}} = 0 \text{ (which means } \mathbf{v}\perp\mathbf{w})
   \end{aligned}$$

3. Using the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ from the previous question, write down a matrix $P$ so that $P\mathbf{i} = \mathbf{u}$, $P\mathbf{j} = \mathbf{v}$ and $P\mathbf{k} = \mathbf{w}$.

   *Solution:* $P$ should have $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as columns.

   $$P = \begin{bmatrix} 5/\sqrt{21} & 0 & 5/\sqrt{105} \\ -2/\sqrt{21} & 1/\sqrt{5} & 8/\sqrt{105} \\ 1/\sqrt{21} & 2/\sqrt{5} & -4/\sqrt{105} \end{bmatrix}$$

Using $P$, write down a product $R$ of 3 matrices so that $R$ represents the transformation of rotation by $\alpha$ about the axis $\mathbf{u}$ (in such a way that $\mathbf{v}$ rotates by $\alpha$ towards $\mathbf{w}$).

*Solution:*

$$
\begin{aligned}
R &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} P^t \\
&= \begin{bmatrix} 5/\sqrt{21} & 0 & 5/\sqrt{105} \\ -2/\sqrt{21} & 1/\sqrt{5} & 8/\sqrt{105} \\ 1/\sqrt{21} & 2/\sqrt{5} & -4/\sqrt{105} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 5/\sqrt{21} & -2/\sqrt{21} & 5/\sqrt{21} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 5/\sqrt{105} & 8/\sqrt{105} & -4/\sqrt{105} \end{bmatrix}
\end{aligned}
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