MA1S12 (Timoney) Tutorial sheet 2
[January 27–31, 2014]

Name: Solutions

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1. Find 3 vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ so that

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 4 & -2 \\ 8 & 9 & 1 \end{bmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Solution: The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ should have components given by the 3 rows of the matrix:

$$\begin{aligned} \mathbf{u} &= \mathbf{i} + 0\mathbf{j} + 3\mathbf{k} \\ \mathbf{v} &= \mathbf{i} + 4\mathbf{j} - 4\mathbf{k} \\ \mathbf{w} &= 8\mathbf{i} + 9\mathbf{j} + \mathbf{k} \end{aligned}$$

How will $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$ be related to the determinant?

Solution: It will be -1 times the determinant (as the change of swapping $\mathbf{u}$ and $\mathbf{w}$ corresponds to swapping two rows of the matrix).

What about $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$?

Solution: This will be the same as the determinant (another swop and so changes the sign back again to the original).

2. A parallelepiped in space has one corner at $(0, 1, 0)$ and adjacent corners at $(1, 2, 3)$, $(2, 3, 4)$ and $(3, 2, 1)$. What is its volume?

Solution: If we give names to the points

$$P = (0, 1, 0), Q = (1, 2, 3), R = (2, 3, 4) \text{ and } S = (3, 2, 1)$$

then the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ are vectors along the 3 edges of the parallelepiped from the corner P. The determinant of the matrix with these are row will give the volume of the parallelepiped (or minus the volume).

$$
\begin{aligned}
\vec{PQ} &= \mathbf{Q} - \mathbf{P} \\
&= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - \mathbf{j} \\
&= \mathbf{i} + \mathbf{j} + 3\mathbf{k} \\
\vec{PR} &= \mathbf{R} - \mathbf{P} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \\
\vec{PS} &= \mathbf{S} - \mathbf{P} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} \\
\text{volume} &= \det \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 1 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -2 \\ 0 & -2 & -8 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & -2 \end{bmatrix} = -(-2)(-2) = -4
\end{aligned}
$$

So the volume is 4.

3. Compute this determinant by cofactor expansion along the second row.

$$
\det \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 1 & 0 \end{bmatrix}
$$

Solution: (The pattern of signs stars with - in the $(2,1)$ place.)

$$
\begin{aligned}
\det \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 1 & 0 \end{bmatrix} &= -0 + \det \begin{bmatrix} 3 & 5 & 6 \\ 0 & 0 & 2 \\ 4 & 1 & 0 \end{bmatrix} - \det \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 2 \\ 4 & 0 & 0 \end{bmatrix} + 0 \\
&= -2 \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} - 4 \det \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} \\
&\quad \text{(expanding first along row 2 and second along row 3} \\
&= 2(3-20) - 4(8-6) = 26
\end{aligned}
$$

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