# Chapter 1. Determinants

This material is in Chapter 2 of Anton & Rorres (or most of it is there). See also section 3.5 of that book.

## 1.1 Introductory remarks

The determinant of a square matrix $A$ is a number $\det(A)$ associated with the matrix $A$, and one of its main properties is that $A^{-1}$ exists exactly when $\det(A) \neq 0$.

Unfortunately the calculation of $\det(A)$, and the explanation of what it is, turns out to be tricky. Certainly it is harder than the trace of $A$. Very vaguely $\det(A)$ is the number you end up dividing by when you compute $A^{-1}$ (and that ties in with the fact that you can't divide by it if it is zero, so that the inverse matrix of $A$ won't make sense if $\det(A) = 0$).

We can make that rather less vague for $2 \times 2$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

In this case you can calculate $A^{-1}$ as a formula. You can do it either by row-reducing

$$[A \mid I_2] = \left[ \begin{array}{cccccc} a_{11} & a_{12} & : & 1 & 0 \\ a_{21} & a_{22} & : & 0 & 1 \end{array} \right]$$

and you should end up with

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Having got this formula somehow, you could also check that it works out. (To do that, multiply the supposed $A^{-1}$ by $A$ to see you do indeed get $I_2$.)

**1.1.1 Definition.** For a $2 \times 2$ matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the *determinant* of $A$ is defined to be the number

$$\det(A) = a_{11}a_{22} - a_{21}a_{21}$$

In words, this is the product of the two diagonal entries minus the product of the two off-diagonal entries.

It is possible to work out a formula for the inverse of a $3 \times 3$ matrix, though it would is quite a bit more messy. There are a number of ways to say what $\det(A)$ is for matrices that are larger than $2 \times 2$. I think there is no one way that is really neat. All the approaches either use a lot of ancillary theory, or else have some significant drawback. The way we will choose now is easy enough to explain, but tricky enough to use as a way of showing that determinants do what they are meant to do. In other words proofs are not so nice when we start the way we are going to do, but we won't really notice that problem because we will skip the proofs!

## 1.2 Cofactor expansion approach to determinants

A quick way to define a determinant is via what is called cofactor expansion along the first row. For $3 \times 3$ matrices this means

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

In words the idea is to multiply each entry of the first row times the determinant of the matrix you get by covering over the first row and the column of the entry. Then add these up with alternating signs $+, -, + \ldots$.

When we start with a $3 \times 3$ matrix $A$, we end up with $\det(A)$ in terms of $2 \times 2$ determinants. And we already know how to evaluate them.

For the $4 \times 4$ case, this idea works out like this

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix} \\ &\quad + a_{13} \det \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} - a_{14} \det \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \end{aligned}$$

You should note the alternating signs and that what we now end up with is 4 determinants of size $3 \times 3$ to calculate. If we expand each of these via cofactors along the first row, we end up with $12 = 4 \times 3$ determinants of size $2 \times 2$ to calculate.

If we use the same approach for $5 \times 5$ determinants, we end up with even more work to do. So this method may be simple enough in principle but it is laborious. We will soon explain a more efficient approach for large matrices.

## 1.3 A formula for determinants

While the above explanation is fine, it is what is called a reduction formula for a determinant. It says how to work out a determinant (of an $n \times n$ matrix $A$ with $n \geq 3$) in terms of smaller determinants. When you keep using the reduction formula enough you get down to $2 \times 2$ determinants and we have a nice tidy formula for them. You might like to have a formula for bigger determinants, not a reduction formula.

Such a thing is available and is described in many standard books (but not in the latest edition of the book by Anton & Rorres). One snag with it is that it requires a bit of theory to explain how it works. I'll outline it below.

In fact the cofactor expansion idea (the reduction formula) works out for $2 \times 2$ determinants as well. If you wanted to use it on a $2 \times 2$ determinant, it would tell you the answer in terms of

$1 \times 1$ determinants! It would say

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \det[a_{22}] - a_{12} \det[a_{21}]$$

and we might have to worry a bit about what the $1 \times 1$ determinants $\det[a_{22}]$ and $\det[a_{21}]$ might mean. Well, a $1 \times 1$ matrix has just one entry, one number in it. If you want to be fussy you can insist that a $1 \times 1$ matrix is a matrix and an number is a scalar, not the same as a matrix. But actually there is rarely a reason to be so fussy and we can usually safely ignore the difference between a $1 \times 1$ matrix and a scalar. The determinant of a $1 \times 1$ matrix is just that scalar. Then the reduction formula works out the right determinant for $2 \times 2$ matrices!

Technically, we should have said what determinants of $1 \times 1$ matrices are and starting with $2 \times 2$ meant that we were not being totally complete. So we've filled in that small detail now, though it is not real important.

Back to a formula for general determinants. If you think about it for a while, it is not hard to see that what you get when you expand out $\det(A)$ completely is a sum of products of entries of $A$ times $\pm 1$. In fact what happens is that, if $A$ is an $n \times n$ matrix, then all products of $n$ entries of $A$ show up which satisfy the restriction that the product contains just one entry from each row of $A$ and one from each column. This is kind of apparent from the cofactor expansion approach. At the beginning we get an entry from the first row times a determinant of a matric where the first row is no longer there (and the column of the entry you have is also no longer present in the smaller determinant).

By arguing in this way, you can establish that what you would get if you multiplied out all the reduction formulae for

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

would be a big sum of terms of the form

$$\pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where $j_1, j_2, \ldots, j_n$ are all of the $n$ column numbers in some order.

So $j_1, j_2, \ldots, j_n$ must be all $n$ of the column numbers $1, 2, \ldots, n$, but not necessarily in that order. In fact all possible orders appear. The possible way to reorder $1, 2, \ldots, n$ are called the permutations of these $n$ numbers. It is possible to see fairly easily that the total number of these permutations is a number called '$n$ factorial'. We write in as $n!$ and it is the product of the numbers $1, 2, \ldots, n$.

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

So this approach gives a formula for the determinant, a formula with $n!$ terms. One snag is that $n!$ gets big pretty fast.

$$3! = 6, \quad 5! = 120, \quad 10! = 3628800$$

So for a $10 \times 10$ matrix this formula would have more than $3.6$ million terms, a lot. Even for $5 \times 5$, you'd have more than 100 terms, each involving a product of 5 terms.

Then there is the problem of which terms get a plus sign and which get a minus. There is a theory about this, and it comes down to something called the 'sign' of a permutation. It would be a digression for us to try and explain what that is in a satisfactory way. So here is a quick explanation. Starting with a permutation

$$j_1, j_2, \ldots, j_n$$

of $1, 2, \ldots, n$, (so that $j_1, j_2, \ldots, j_n$ are all the first $n$ whole numbers written in some order), we are going to write down a matrix called the matrix for this permutation. In each row (and column) the permutation matrix has just one single entry equal to 1, all the others are 0. To be specific, in row number $i$, there is a 1 in column $j_i$, and zeros elsewhere. (Another way to say it is that there are entries $= 1$ at the positions $(i, j_i)$ for $i = 1, 2, \ldots, n$, but every other entry is 0.) The sign of the permutation is the same as the determinant of its permutation matrix.

Well, that is a true statement, but it is a bit unsatisfactory. Our long formula for a determinant still has some determinants in it, the ones that give the $\pm 1$ signs.

There is a way to say how the whole formula works out for $3 \times 3$ matrices, and it is a fairly satisfactory way of working out $3 \times 3$ determinants. The drawback is that it does not extend to bigger determinants in any very similar way.

Starting with a $3 \times 3$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

write it down with the first two columns repeated

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Add the products diagonally to the right and subtract those diagonally to the left as indicated by the arrows

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\qquad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

Notice that there are the correct number of terms here ($3! = 6$). And each product of 3 has one entry from each row, one entry from each column.

As mentioned above, this method of repeating the first and second columns does **not** work for sizes apart from $3 \times 3$, and there is nothing really like this for $4 \times 4$ or bigger matrices. The cofactor expansion method does work for any size of (square) matrix.

## 1.4 Properties of determinants

Here are the key properties of determinants. We'll explain why they are true in the case of $2 \times 2$ determinants, and give short shift to the explanations of why these properties still work for $n \times n$ determinants.

(i) $\det(I_2) = 1$

This is pretty easy to see.

(ii) $\det(AB) = \det(A)\det(B)$

*Proof.* We should show that this is true for any $2 \times 2$ matrices $A$ and $B$, without knowing what the entries are in the matrices. What we do is write out the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Then multiply out

$$\det(AB) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

and

$$\det(A)\det(B) = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

to show that the answers are the same.

It is not really hard to do, though maybe not profitable to write the remaining steps out for you. You might like to convince yourself that it does work out as claimed. Doing it for yourself is more likely to convince you that it works. □

(iii) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

*Proof.* Using the previous result

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I_2) = 1$$

and so $\det(A^{-1}) = 1/\det(A)$. □

(iv) $\det(A^t) = \det(A)$

This is not at all hard (for the $2 \times 2$ case).

(v) The determinants of elementary matrices are as follows

$1'$. $E$ the elementary matrix for the row operation "multiply row 1 by $k \neq 0$"
$E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\det(E) = k$.

1″. $E$ the elementary matrix for the row operation "multiply row 2 by $k \neq 0$"
$E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $\det(E) = k$.

2′. $E$ the elementary matrix for the row operation "add $k$ times row 2 to row 1"
$E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $\det(E) = 1$.

2″. $E$ the elementary matrix for the row operation "add $k$ times row 1 to row 2"
$E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $\det(E) = 1$.

3. $E$ the elementary matrix for the row operation "swop rows 1 and 2"
$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det(E) = -1$.

For general $n \times n$ determinants, all these statements remain true, when suitably phrased.

**1.4.1 Lemma.** *The determinants of elementary matrices are as follows*

1. *$E$ the elementary matrix for the row operation "multiply a row by $k \neq 0$" has* $\det(E) = k$.
2. *$E$ the elementary matrix for the row operation "add $k$ times one row to a different row" has* $\det(E) = 1$.
3. *$E$ the elementary matrix for the row operation "swop two specified rows" has* $\det(E) = -1$.

This leads us to a way to calculate $n \times n$ determinants. Starting with the matrix $A$, do row operations on $A$ to row reduce $A$. At the first row operation we are replacing the matrix $A$ by $EA$ for some elementary matrix $A$. So the determinant of the matrix we have changes to

$$\det(EA) = \det(E)\det(A)$$

We've seen that $\det(E)$ is easy to figure out. So it is quite easy to keep track of the changes in the determinant as we do each row operation. We'll organise this better next.

If we keep doing row operations, keeping track of how the determinant changes as we go along, we will get to an upper triangular matrix and we can say this:

**1.4.2 Lemma.** *If $A$ is upper triangular or if $A$ is lower triangular then* $\det(A)$ *is the product of the diagonal entries.*

*Proof.* For a lower triangular matrix like

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{32} & 0 \\ a_{41} & a_{42} & a_{42} & a_{44} \end{pmatrix}$$

it is quite easy to see that expanding along the first row gives $a_{11}$ times some determinant (in fact the determinant of a lower triangular $3 \times 3$ matrix in this case) plus zero. An expanding further we just get the product of the diagonals.

For the upper triangular case we could relay on the fact that the transpose will be lower triangular (and has the same determinant and the same diagonal entries). So for example

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} = \det \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = a_{11}a_{22}a_{33}a_{44}$$

This is not a fully correct proof as we have not dealt with arbitrary sized matrices, but it should be enough to explain the idea for the general case. $\square$

**1.4.3 Theorem.** *(Simplification rules and properties of determinants) Let $A$ be an $n \times n$ matrix throughout.*

*(i) Adding a multiple of one row of $A$ to another row results in a matrix with unchanged determinant.*

*(ii) Factoring a scalar $k \neq 0$ out from a single row of $A$ divides the determinant by $k$. That is*

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = k \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{i1}/k & a_{i2}/k & \cdots & a_{in}/k \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

*(iii) Swapping two rows of $A$ changes the determinant by a factor $-1$.*

*(iv) An efficient way to work out $\det(A)$ is to use Gaussian elimination to row reduce $A$ to row-echelon form, keeping track of how the determinant changes after each row operation (see properties (i) – (iii)). Notice that the row-echelon form will be upper triangular with either all 1's on the diagonal or some diagonal entries are 0 (so that the determinant of the row-echelon form is 1 in the case where the diagonal entries are all 1, which happens when $A$ is invertible, 0 othewise).*

*Proof.* The first 3 rules follow from what we said already about determinants of products and determinants of elementary matrices.

For (ii) it may be a bit confusing to see it, but notice that

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = E \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{i1}/k & a_{i2}/k & \cdots & a_{in}/k \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where $E$ is the diagonal elementary matrix with $k$ at position $(i, i)$ and 1's in the other diagonal places (the elementary matrix for the row operation that multiplies row $i$ across by $k$).

As for the efficiency claim, working out the determinant directly by expanding will typically take around $n!$ (multiplication) operations, whereas Gaussian elimination will take more like $n^3/3$. We mentioned already that $10!$ is $3.6$ million and $10^3$ is 'only' a thousand. For $n$ that is any way large $n!$ is prohibitive, while $n^3/3$ is much smaller. $\square$

**1.4.4 Examples.** 1. Find $\det \begin{bmatrix} 0 & 1 & 5 \\ 1 & 2 & 4 \\ 4 & 5 & 7 \end{bmatrix}$ via row reduction.

$$\begin{aligned}
\det \begin{bmatrix} 0 & 1 & 5 \\ 1 & 2 & 4 \\ 4 & 5 & 7 \end{bmatrix} &= -\det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & -3 & -9 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix} \\
&= -6 \det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = -6
\end{aligned}$$

2. Show $\det \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \neq 0$ if $x$, $y$ and $z$ are all different.

(Notice that the determinant would be zero if any two of $x$, $y$ and $z$ were equal. In this case the matrix would have two identical columns, and so determinant zero.)

To solve this we will first transpose the matrix and then use row operations on the transpose.

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} &= \det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 1 & z & z^2 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{bmatrix} \\
&= (y-x)\det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & z-x & (z-x)(z+x) \end{bmatrix} \\
&= (y-x)(z-x)\det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{bmatrix} \\
&= (y-x)(z-x)\det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z+x-(y+x) \end{bmatrix} \\
&= (y-x)(z-x)\det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{bmatrix} \\
&= (y-x)(z-x)(z-y)
\end{aligned}
$$

If $x$, $y$ and $z$ are all different, then all 3 factors in the determinant are different from 0. So their product is not zero.

By the way, this determinant we just worked out has a special name. It is called a *Vandermonde determinant*.

**1.4.5 Theorem.** *If $A$ is an $n \times n$ matrix, then the following are two equivalent statements about $A$:*

*(a)* *$A$ is invertible*

*(b)* $\det(A) \neq 0$.

This can then be added as an extra part to a theorem you have seen before. Recall that there are several ways to recognise invertible matrices:

*Let $A$ be an $n \times n$ (square) matrix.*

*The following are equivalent statements about $A$, meaning that is any one of them is true, then the other have to be true as well. (And if one is not true, the others must all be not true.)*

*(a)* *$A$ is invertible (has an inverse)*

*(c)* *the equation $A\mathbf{x} = \mathbf{0}$ (where $\mathbf{x}$ is an unknown $n \times 1$ column matrix, $\mathbf{0}$ is the $n \times 1$ zero column) has only the solution $\mathbf{x} = \mathbf{0}$*

*(d)* *the reduced row echelon for $A$ is $I_n$*

*(e)* *$A$ can be written as a product of elementary matrices*

*(f)* *there is an $n \times n$ matrix $B$ with $BA = I_n$*

*(g)* *there is an $n \times n$ matrix $B$ with $AB = I_n$*

*Proof.* If you recall the equivalence of (a) and (d) [something discussed last semester and so we take it as known] we can deduce the equivalence of (a) and (b).

In fact we have that (a) implies (b) because of the fact that if $A$ has an inverse then $\det(A^{-1}) = 1/\det(A)$ — so that $\det(A) \neq 0$. So if we know (a) is true about $A$, then (b) must also be true.

On the other hand suppose we know that (b) holds, that is that $\det(A) \neq 0$. If we row reduce $A$, the determinant may change by various factors for some of the row operations, but the factors are always nonzero. So when we end up with a row echelon form (triangular) the diagonal entries must be all 1 (otherwise we would have zero for $\det(A)$). If we continue with Gauus-Jordan eleimination to reduced row echelon form we must get the identity matrix. So $A$ satisfies (d), hence satisfies (a). $\square$

## 1.5 Cross products (recap.)

These were already discussed in MA1S11.

This is something that makes sense in three dimensions only. There is no really similar product of two vectors in higher dimensions. In this respect it is different.

For example, we might have started with dot products in 2 dimensions, then extended the notion to 3 dimensions, and later realised that the formula we had for dot products has an obvious extension to 4 dimensions $\mathbb{R}^4$, to $\mathbb{R}^5$ and to every $\mathbb{R}^n$. We just need to extend the formula in a rather easy way. Most of the other formulae we had also extend to $\mathbb{R}^n$ with no real bother. Cross products are different. (By the way, this topic is in §3.5 of Anton & Rorres, as is the material below about areas and volumes.)

The easiest way to remember the formula for the *cross product* $\mathbf{v} \times \mathbf{w}$ of two vectors

$$\begin{aligned}
\mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \\
\mathbf{w} &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k},
\end{aligned}$$

in space is to use a kind of "determinant" formula

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

Now we have not allowed determinants of matrices where the entries are not scalars, and we should not allow it. However, in this particular case, we can get away with it if we interpret the determinant as what we would get by a cofactor expansion along the first row. So, in a more correct statement, the definition of the cross product is

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \mathbf{i} \det \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \\ &= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_2)\mathbf{k} \\ &= (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_2)\mathbf{k} \end{aligned}$$

There is actually a pattern[1] to this last formula and so it is not quite impossible to remember. But the (slightly suspect) determinant formula is easier to recall, I think.

## 1.6 Properties of cross products (in $\mathbb{R}^3$)

Again this was covered in MA1S11.

(i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.

(ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$

(iii) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(iv) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$ where $\theta =$ the angle between $\mathbf{v}$ and $\mathbf{w}$.

(v) Now that we know in a geometrical way the length of $\mathbf{v} \times \mathbf{w}$, and we also know that it is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, we can describe cross products in a geometrical way.

If the angle $\theta$ between $\mathbf{v}$ and $\mathbf{w}$ is not 0 and not $\pi$, then the vectors $\mathbf{v}$ and $\mathbf{w}$ are not in the same direction and also not in exactly opposite directions. So as long as $0 < \theta < \pi$, then we can say that there is one plane through the origin parallel to both $\mathbf{v}$ and $\mathbf{w}$ (or containing both vectors if we draw them from the origin). The cross product is then in one of the two normal directions to that plane.

If $\theta = 0$ or $\theta = \pi$, there is no one plane containing $\mathbf{v}$ and $\mathbf{w}$, but in these cases $\sin\theta = 0$ and so we know $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

[1] The first component of $\mathbf{v} \times \mathbf{w}$ depends on the components of $\mathbf{v}$ and $\mathbf{w}$ other than the first. Starting with $v_2w_3 - v_3w_2$ we can get to the next component by adding 1 to the subscripts and interpreting $3 + 1$ as 1. Or think in terms of cycling the subscripts around $1 \to 2 \to 3 \to 1$ to get the next component. You still have to remember the first one.

In the case $0 < \theta < \pi$, we can describe the cross product up to one of the two normal directions to the plane. The question then is to say which direction it is in. If we can identify the top (or upwards) side of the plane somehow, is the cross product pointing up or down? And if the plane is vertical? The answer to this depends on having the axes fixed in such a way that the direction of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey a 'right-hand rule'. This can be described in terms of the directions of the index finger, first finger and thumb on your right hand if you hold them perpendicular to one another. Another way is to place a corkscrew (an ordinary right-handed corkscrew) along the vertical axis and twist the screw from the $x$-axis towards the $y$-axis. It should travel in the direction of the positive $z$-axis.

For two vectors $\mathbf{v}$ and $\mathbf{w}$, the direction of $\mathbf{v} \times \mathbf{w}$ is described by a right-hand rule. Imaging a corkscrew placed so it is perpendicular to the plane of $\mathbf{v}$ and $\mathbf{w}$. Turn the screw from $\mathbf{v}$ towards $\mathbf{w}$ and the direction it travels is the same as the direction of $\mathbf{v} \times \mathbf{w}$.

(vi) There are some algebraic properties of the cross product that are as you would expect for products:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \\ (\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \\ (k\mathbf{v}) \times \mathbf{w} &= k(\mathbf{v} \times \mathbf{w}) \\ &= \mathbf{v} \times (k\mathbf{w}) \end{aligned}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and any scalar $k \in \mathbb{R}$. (But recall that the order matters since $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.)

(vii) Scalar triple products. (Again mentioned in MA1S11.) We can take the dot product of a thrd vector $\mathbf{u}$ with the cross product $\mathbf{v}) \times \mathbf{w}$. We get (a scalar value)

$$\begin{aligned} &\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \\ &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot \left( \mathbf{i} \det \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) \\ &= u_1 \det \begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - u_2 \det \begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + u_3 \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \\ &= \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \end{aligned}$$

**1.6.1 Example.** Find the equation of the plane that goes through $(1, 2, 3)$, $(3, 1, 2)$ and $(2, 3, 1)$.
*Solution:* There are other ways, but here is an approach using cross products.

Let $P = (1, 2, 3)$, $Q = (3, 1, 2)$ and $R = (2, 3, 1)$ and use the same letters for the position vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. Then we can notice that

$$\begin{aligned}
\vec{PQ} &= \mathbf{Q} - \mathbf{P} \\
&= (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\
&= 2\mathbf{i} - \mathbf{j} + \mathbf{k} \\
\vec{PR} &= \mathbf{R} - \mathbf{P} \\
&= (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} - 2\mathbf{k}
\end{aligned}$$

are two vectors that are in the plane we want (or parallel to it). So their cross product must be normal to the plane:

$$\begin{aligned}
\vec{PQ} \times \vec{PR} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\
&= \mathbf{i} \det \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

So the plane we want has an equation

$$3x + 3y + 3z = \text{ const.}$$

and we can plug in any of the points $P$, $Q$ or $R$ to see that the constant has to be 18. Thus the equation of the plane is

$$3x + 3y + 3z = 18$$

or rather this is one possible equation. We can multiply or divide this equation by any nonzero number and still have an equation for the plane. A tidier-looking equations is

$$x + y + z = 6$$

(In retrospect maybe we could have guessed the equation, because the 3 points $P$, $Q$ and $R$ had the same coordinates permuted around. But the method we used would work for any 3 points, as long as they did not lie in a line.)

## 1.7 Geometrical view of determinants

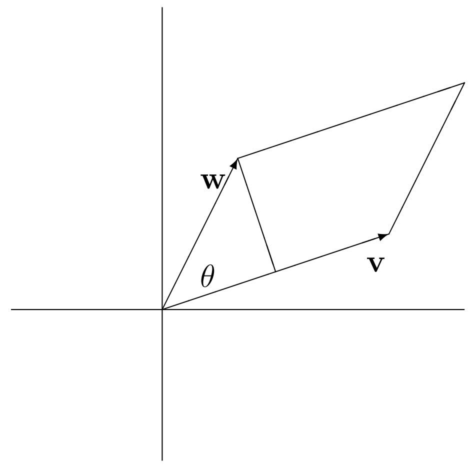
For $2 \times 2$ matrices and $3 \times 3$ matrices, there is a graphical interpretation of the determinant.

Consider the case of $2 \times 2$ matrices first, and think of the rows of the matrix as components of vectors in the plane $\mathbb{R}^2$.

$$\det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$$

So we are thinking of the two vectors $\mathbf{v} = (v_1, v_2) = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = (w_1, w_2) = w_1\mathbf{i} + w_2\mathbf{j}$. If we draw the two vectors from the origin (the position vectors of the points $(v_1, v_2)$ and $(w_1, w_2)$) and fill in the parallelogram that has $\mathbf{v}$ and $\mathbf{w}$ as sides, then

$$\det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} = \pm(\text{the area of the parallelogram})$$




The area of that parallelogram is

$$\text{base } \times \text{ perpendicular height} = \|\mathbf{v}\|(\|\mathbf{w}\| \sin\theta)$$

We could work out $\sin\theta$ by first working from $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$, and then using $\sin^2\theta + \cos^2\theta = 1$. Here is an alternative way that needs less work if we allow ourselves to think in 3 dimensions for a bit.

If we think of $\mathbf{v}$ and $\mathbf{w}$ as vectors in space that happen to have zero in their 3rd components,

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + 0\mathbf{k} \text{ and } \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + 0\mathbf{k},$$

then we can calculate their cross product and we get

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & 0 \\ w_1 & w_2 & 0 \end{bmatrix} = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \mathbf{k}$$

But we know

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta = \text{ the area of the parallelogram}$$

and from the calculation of the cross product its length $\|\mathbf{v} \times \mathbf{w}\|$ is the absolute value of the determinant. So

$$\boxed{\text{the area of the parallelogram} = \pm \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}}$$

If we think a bit harder it is possible to work out which sign the determinant has. If the determinant is positive, it means that the corss product is in the positive $z$-direction (or in the same direction as $\mathbf{k}$). From the right-hand rule for cross products, that means that whne you turn from $\mathbf{v}$ toward $\mathbf{w}$ (in the angle that is smaller than $\pi$) you are turning anticlockwise.

Of course there is a case where the angle is $\pi$ but in that case the cross product has length zero ($\sin\theta = 0$) and the 'parallelogram' is not really a parallelogram, but a line with no area. That also happens if $\theta = 0$ (the two vectors are in the same direction).

If you feel unhappy that the exmplanation just give uses 3 dimensional ideas, there is another way to work stuff out using just two dimensions.

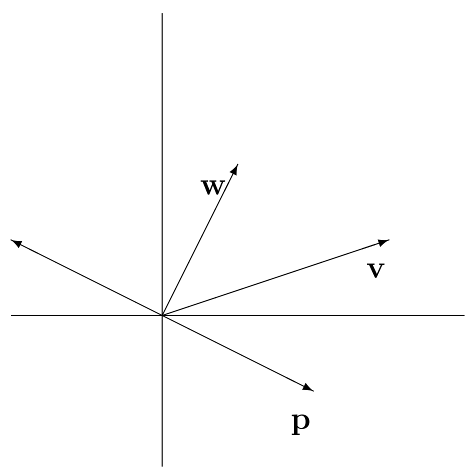
We know $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$. The vector

$$\mathbf{p} = w_2\mathbf{i} - w_1\mathbf{j}$$

has the same length $\|\mathbf{p}\| = \sqrt{w_2^2 + w_1^2} = \|\mathbf{w}\|$ as $\mathbf{w}$ and it is perpendicular to $\mathbf{w}$ because

$$\mathbf{p}\cdot\mathbf{w} = w_2w_1 - w_1w_2 = 0.$$

There are two vectors in the plane with that length and direction perpendicular to $\mathbf{w}$.




In our picture, the one that makes an angle $(\pi/2) - \theta$ with $\mathbf{v}$ is $\mathbf{p}$ (but that depends on the fact that $\mathbf{w}$ is $\theta$ radians anticlockwise from $\mathbf{v}$ — if it was clockwise then the right one would be $-\mathbf{p}$ instead).

So we have

$$\mathbf{v}\cdot\mathbf{p} = \|\mathbf{v}\|\|\mathbf{p}\|\cos\left(\frac{\pi}{2} - \theta\right) = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

and so

$$v_1w_2 - v_2w_1 = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta = \text{ area of parallelogram}$$

$$\det\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} = \text{ the area of the parallelogram}$$

As we mentioned, we don't always get this with a plus sign. Sometimes we get a minus (when $\mathbf{w}$ is clockwise from $\mathbf{v}$).

Notice that if $\mathbf{v}$ is parallel to $\mathbf{w}$, or if they are in exactly opposite directions, the parallelogram collapses to a line, and so we get zero area. In this case one row of the matrix is a multiple of the other.

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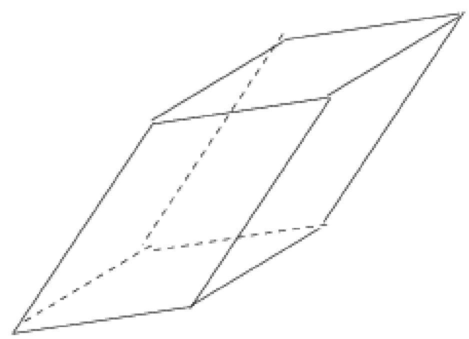
When we move to 3 dimensions, we can get a similar result, but we need a three dimensional version of a parallelogram. Three vectors in space will span a *parallelepiped*. Its a shape like a box without right angles between the sides.



All the faces are parallelograms, and opposite faces are parallel to one another. If we think of the origin as one corner, the 3 sides emanating from there could represent three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in space.

On the other hand if we start with 3 vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, then we can draw them all as arrows starting from the origin. Then we can make a parallelepiped using those as three of the edges.

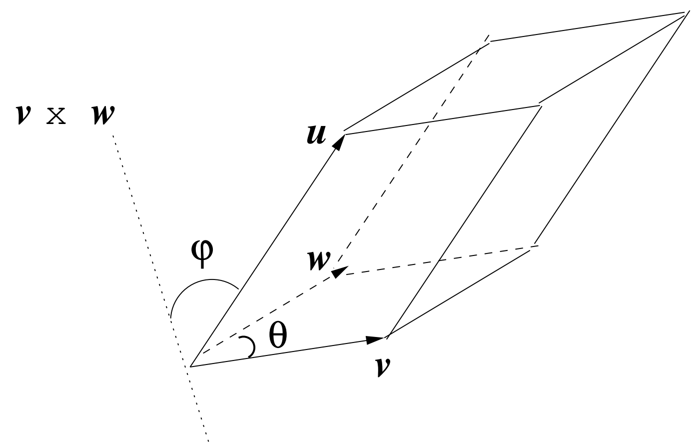
The link with determinants is

$$\boxed{\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \pm(\text{volume of the parallelepiped}) \qquad (1)}$$

## 1.8 Volume of a parallelepiped

We'll now give a reason (with the help of cross products) why $3 \times 3$ determinants can be interpreted as volumes of parallelepipeds. That is we will justify equation (1) now.

As before we consider the parallelepiped determined by the 3 vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$.




We take $\theta$ for the angle between $\mathbf{u}$ and $\mathbf{v}$ and look at the face spanned by these vectors as the 'base' of the parallelepiped. The volume is given by a similar formula to the one for the area of a parallelogram

$$\text{volume (parallelepiped)} = \text{ area of base } \times \text{ perpendicular height}$$

Geometrically we know

$$\text{area of base } = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$$

where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$. But we can also recognise this as the length of the cross product and so

$$\text{area of base } = \|\mathbf{v} \times \mathbf{w}\|$$

The direction of the cross product $\mathbf{v} \times \mathbf{w}$ is perpendicular to the base, and in our picture it is in the 'upwards' direction, though it could be going the opposite way. Take $\phi$ for the angle between $\mathbf{u}$ and the direction of $\mathbf{v} \times \mathbf{w}$. So the perpendicular height is

$$\text{perpendicular height } = \pm\|\mathbf{u}\| \cos\phi$$

(where the minus sign would be needed if $\cos\phi < 0$ and the cross product was in the 'downwards' direction from the base). We usually compute angles between vectors using the dot product and so we should look at

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \|\mathbf{v} \times \mathbf{w}\|\|\mathbf{u}\| \cos\phi \\
&= (\text{ area of base }) \; (\pm(\text{ perpendicular height })) \\
&= \pm \text{ volume (parallelepiped)}
\end{aligned}$$

We have already computed that the expression $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (the scalar triple product) comes out as a determinant. So we end up with

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \pm \text{ volume (parallelepiped)}$$

as we claimed.

We could also try to figure out whether the sign is plus or minus. That would depend on whether the cross product make an acute angle $\phi$ with $\mathbf{u}$ or an obtuse angle (and has to do with a kind of right-handedness or left-handedness of the way the sides $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ are arranged.

So we have shown now that the determinant if $\pm$ the volume (and we also showed that a scalar triple product is a determinant, though that makes sense only for vectors in 3 dimensions).

In a way it is true that determinants of $n \times n$ matrices give some sort of $n$-dimensional volume. We've proved it in 2 dimensions (parallelogram case) and in 3 dimensions (parallelepiped), but the idea of an $n$-dimensional volume is beyond us in this course.

**1.8.1 Example.** Find the volume of the parallelepiped in space where one corner is at $(0, 0, 0)$ and the 3 vectors

$$\begin{aligned} \mathbf{u} &= 4\mathbf{i} - 2\mathbf{j} + \mathbf{k} \\ \mathbf{v} &= 5\mathbf{i} + \mathbf{j} - \mathbf{k} \\ \mathbf{w} &= \mathbf{i} - \mathbf{j} + \mathbf{k} \end{aligned}$$

are parallel to three of the edges.

**Answer** is

$$\det \begin{bmatrix} 4 & -2 & 1 \\ 5 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \pm(\text{volume})$$

So the absolute value of the determinant gives the volume.

$$\begin{aligned} \det \begin{bmatrix} 4 & -2 & 1 \\ 5 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} &= 4 \det \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 5 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= 4 \det \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{7}{2} & -\frac{9}{4} \\ 0 & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \\ &= 4 \left(\frac{7}{2}\right) \det \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & -\frac{9}{14} \\ 0 & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \\ &= 14 \det \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & -\frac{9}{14} \\ 0 & 0 & \frac{3}{4} - \frac{9}{28} \end{bmatrix} \\ &= 14 \left(-\frac{3}{7}\right) = -6 \end{aligned}$$

So the volume is 6.

## 1.9 A further look at properties of determinants

We can use the scalar triple product formula

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

to prove some more properties of determinants (admittedly only for the $3 \times 3$ case) and also to redo some of those we gave earlier in Theorem 1.4.3 (where the indicated proofs depended on knowing that $\det(AB) = \det(A)\det(B)$).

We can see from the scalar triple product that exchanging $\mathbf{v}$ and $\mathbf{w}$ (or rows 2 and 3 of the matrix) will change the determinant by a factor $-1$ because $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$. It should also be true that exchanging $\mathbf{u}$ with one of $\mathbf{v}$ or $\mathbf{w}$ (exchanging the first row with one of the others) will also change the determinant by $-1$, but that does not seem so obvious.

However, it is clear that if $\mathbf{u} = \mathbf{v}$ then the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot (\mathbf{u} \times \mathbf{w}) = 0$ (one of the properties of cross products). So the determinant is zero if rows 1 and 2 are equal (or in fact if any two rows are equal if you think a little about how cross products work).

Then we have

$$(\mathbf{u} + \mathbf{v}) \cdot ((\mathbf{u} + \mathbf{v}) \times \mathbf{w}) = 0$$

and we can expand that out to get

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = 0$$

(two other terms vanish).

Here now comes a theorem listing properties of determinants, some repetitive of ones we saw earlier.

**1.9.1 Theorem.** *Let $A$ be an $n \times n$ matrix throughout.*

*(i)* *If $A$ has a row of all zeros, then* $\det(A) = 0$.

*(ii)* *If $A$ has two rows where one row is a multiple of the other, then* $\det(A) = 0$.

*(iii)* *$A$ is invertible exactly when* $\det(A) \neq 0$. *(That's Theorem 1.4.5.)*

*(iv)* $\det(AB) = \det(A)\det(B)$ *if $B$ is $n \times n$. (We have already been using this! We have really only proved it for the $2 \times 2$ case but we won't do any more about that.)*

*(v)* $\det(A^{-1}) = 1/\det(A)$ *if $A$ is invertible. (Again we saw this already and the proof we gave earlier was fine for the general case.)*

*(vi)* $\det(A^t) = \det(A)$. *(Also mentioned before.)*

*(vii)* *There is another efficient way to work out determinants, similar to (iv) of Theorem 1.4.3, where one uses 'column operations' instead of row operations.*

*Proof.* We will not try to justify most of this, relying instead on the previous explanations for the $2 \times 2$ case or the above explanations in the $3 \times 3$ case.

We did not mention (i) or (ii) previously. (i) is fairly obvious when the matrix is small, and even for general matrices if you think of the expansion of the determinant into a sum of $n!$ terms. Each of the $n!$ terms will be zero because it has to have an entry from the row of all zeroes.

(ii) follows because we can subtract one of the two identical row from the other without changing the deteminant — and then we have a row of all zeroes.

The last statement (vii) can be justified by transposing the matrix. Column operations on the original matrix became row operations on the transpose.

It might well cause confusion to start trying to use column operations and so we will instead take the transpose and do row oerations — that Vandermonde example was already of this kind. □

## 1.10 A determinant formula for the inverse matrix

If $A$ is an $n \times n$ matris, then we know more or less that $\det(A)$ is the number that you have to dvide by to calculate $A^{-1}$. We can now give a formula that makes that very clear.

**1.10.1 Definition** (Matrix of cofactors)**.** If $A$ is an $n \times n$ matrix with $(i,j)$ entry $a_{i,j}$ we define the *cofactor for the* $(i,j)$ *entry* to be

$$C_{ij} = (-1)^{i+j} \det(M_{i,j})$$

where $M_{i,j}$ is the matrix you get by covering over row $i$ and column $j$ of $A$. (So the matrices $M_{i,j}$ are $(n-1) \times (n-1)$ matrices.)

The *matrix of cofactors of* $A$ is the $n \times n$ matrix with $(i,j)$ entry $C_{i,j}$.

**1.10.2 Remark.** *In this terminology we can say that the way we defined the determinant was*

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}$$

*That is multiply the entries in the first row times their cofactors, and add up the results.*

*The signs* $(-1)^{i+j}$ *that go on the cofactors start with plus in the top left corner and alternate their signs along rows or down columns, like this*

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

**1.10.3 Definition** (Adjugate matrix)**.** If $A$ is an $n \times n$ matrix, then the *adjugate matrix of* $A$ is the transpose of the matrix of cofactors. We will denote this matrix by $\mathrm{adj}(A)$.

**1.10.4 Theorem** (Determinant formula for the inverse)**.** *If* $A$ *is an* $n \times n$ *matrix, then*

$$A\,\mathrm{adj}(A) = \det(A) I_n$$

*From this we see that*

*(i) If* $\det(A) \neq 0$*, then*

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

*(ii) We can expand* $\det(A)$ *by cofactor expansion along any row because multiplying row* $i$ *of* $A$ *times column* $i$ *of* $\operatorname{adj}(A)$ *gives*

$$a_{1,1}C_{1,i} + a_{i,2}C_{i,2} + \cdots + a_{1,n}C_{1,n} = \det(A)$$

*(iii) We can expand* $\det(A)$ *by cofactor expansion along any column.*

*Proof.* We won't make any attempt to prove that $A \operatorname{adj}(A) = \det(A) I_n$.

The rest are really more or less immediate consequences. If $\det(A) \neq 0$ then we can say

$$A\left(\frac{1}{\det(A)} \operatorname{adj}(A)\right) = I_n$$

That says that $(1/\det(A)) \operatorname{adj}(A)$ is the inverse. So the product works the other way around also

$$\left(\frac{1}{\det(A)} \operatorname{adj}(A)\right) A = I_n$$

and therefore $\operatorname{adj}(A)A = \det(A)I_n$.

The formula $A \operatorname{adj}(A) = \det(A)I_n$ does imply that row $i$ of $A$ times column $i$ of $\operatorname{adj}(A)$ gives $\det(A)$ (the $(i,i)$ entry of $\det(A)I_n$. Since $\operatorname{adj}(A)$ is the transpose of the cofactor matrix, column $i$ of $\operatorname{adj}(A)$ is row $i$ of the cofactor matrix.

If you use $\operatorname{adj}(A)A = \det(A)I_n$ and work out what that says for the $(i,i)$ entry, you end up with the cofactor expansion along column $i$. □

**1.10.5 Example.** In the case of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of cofactors (just a matrix of $1 \times 1$ determinants with $\pm$ signs attached is

$$\text{cofactor matrix}(A) = \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}$$

and taking the transpose we get the adjugate

$$\operatorname{adj}(A) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So we get again the familiar formula for the inverse of a $2 \times 2$ matrix

$$A = \frac{1}{\det(A)} \operatorname{adj} A = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**1.10.6 Example.** Find $\operatorname{adj}(A)$ when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

*Solution:* The cofactor matrix for $A$ is

$$\begin{bmatrix} +\det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} & -\det \begin{bmatrix} 0 & 5 \\ 0 & 6 \end{bmatrix} & +\det \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \\ -\det \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} & +\det \begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix} & -\det \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ +\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} & -\det \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} & +\det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 24 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 4 \end{bmatrix}$$

Take the transpose to get the adjugate:

$$\operatorname{adj}(A) = \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

(We could verify that $A \operatorname{adj}(A) = \det(A) I_3$. In fact $\det(A) = 24$ in this case because $A$ is triangular and so the determinant is the product of the diagonal elements (of $A$). We also have $A^{-1} = (1/\det(A)) \operatorname{adj}(A) = (1/24) \operatorname{adj}(A)$ and $A^{-1}$ is upper trainangular — as you should know would be the case since $A$ is upper triangular.)

## 1.11 Cramer's rule

This is a topic that is treated in many textbooks, a fairly direct consequence of the formula $A^{-1} = (1/\det(A)) \operatorname{adj}(A)$ for the inverse.

We recall that if we have a system of $n$ equations in $n$ unknowns, then it can be written as a single matrix equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where $A$ is the matrix of coefficients in the left hand sides of the equations, $\mathbf{x}$ are the $n$ unknowns written as a column matrix ($n \times 1$ column) and $\mathbf{b}$ is the right-hand sides again written as a column.

More precisely stated, the idea is that a system of $m$ linear equations in $n$ unknowns

$$\left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right.$$

can be written as a single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

which is of the form $A\mathbf{x} = \mathbf{b}$ we mean. However, I am concentrating now on the situation where $m = n$ and $A$ is a square matrix.

Now recall that in the case of the same number of equatioins as unknowns, **if we know** that the (square) matrix $A$ has an inverse, and we know what $A^{-1}$ is, then we can easily solve $A\mathbf{x} = \mathbf{b}$ by multiplying both sides on the lect by $A^{-1}$ to get $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ or

$$\mathbf{x} = A^{-1}\mathbf{b}$$

We can combine that with our formula for the iverse to get

$$\mathbf{x} = \frac{\operatorname{adj}(A)\mathbf{b}}{\det(A)}$$

and further we can express the result in terms of $n \times n$ determinants. To calculate $\operatorname{adj}(A)\mathbf{x}$ we multiply the rows of $\operatorname{adj}(A)$ times the column $\mathbf{b}$. But the rows of $A$ are the columns of the cofactor matrix. So, for instance the first unknown $x_1$ is

$$x_1 = \frac{C_{1,1}b_1 + C_{2,1}b_2 + \cdots + C_{n,1}b_n}{\det(A)}$$

The top line of this fraction is actually a determinant also (expanded along column 1 via cofactor expansion). We have

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}}{\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}$$

The denominator is the determinant of the matrix $A$ of the coefficients, while the determinant in the numerator has the first column replaced by the column $\mathbf{b}$ of right hand sides.

What Cramer's rule says is that

$$x_j = \frac{\det(\text{matrix got from } A \text{ by replacing column } j \text{ by } \mathbf{b})}{\det(A)}$$

**1.11.1 Example.** Write a determinant formula for $x_3$ if $(x_1, x_2, x_3, x_4)$ solves the system

$$\begin{cases} 5x_1 & + & 8x_2 & - & 2x_3 & + & x_4 & = & -9 \\ & & x_2 & + & 4x_3 & - & 7x_4 & = & 3 \\ & & & & 11x_3 & + & 2x_4 & = & -1 \\ x_1 & + & x_2 & - & 8x_3 & + & 3x_4 & = & 0 \end{cases}$$

(assuming that the matrix of coefficients is invertible).

*Solution:* By Cramer's rule we have

$$x_3 = \frac{\det \begin{bmatrix} 5 & 8 & -\mathbf{9} & 1 \\ 0 & 1 & \mathbf{3} & -7 \\ 0 & 0 & -\mathbf{1} & 2 \\ 1 & 1 & \mathbf{0} & 3 \end{bmatrix}}{\det \begin{bmatrix} 5 & 8 & -2 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 11 & 2 \\ 1 & 1 & -8 & 3 \end{bmatrix}}$$

(Note that column 3 on the top is the one that is different from the one below. The determinant below is the determinat of the matrix of coefficients. On the top column 3 is replaced by the column of right hand sides.)

## 1.12 Using determinants to find equations

In this section we present a kind of application of determinants. Not exactly a scientific application, as we will just be looking for equations. To illustrate the method, we'll start with a complicated way of finding the equation of a line through 2 points in the plane, then go on to the equation of a circle through 3 specified points, and finally look at the equation of a plane in space through 3 given points. In most of these examples we can already do them another way, and the determinant approach is kind of neat from the point of view of a mathematician. Not so really practical. Something like this is explained in §10.1 of Anton & Rorres.

**Line through 2 points in the plane.** If we are given two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in the plane we already know several ways to find the line going through them.

If the points are not on a vertical line (which would have to have equation $x = p_1$ in the case $p_1 = q_1$) then the line has a slope

$$m = \frac{q_2 - q_1}{p_2 - p1}$$

and then it has to have an equation like $y = mx + c$. We can find $c$ now that we know $m$ by plugging in either $P$ or $Q$ (into $y = mx + c$). Maybe even slicker is to write down

$$y - p_2 = m(x - p_1)$$

as the equation, or even

$$y - p_2 = \left(\frac{q_2 - q_1}{p_2 - p1}\right)(x - p_1)$$

So we certainly don't need determinants to do this, but we are just using it as a warm-up example. The equation of a line is either $x = c$ if it is vertical or $y = mx + c$ if it has a slope. To cover both cases at once we can write

$$ax + by = c$$

or

$$ax + by - c = 0$$

The coefficients $a$, $b$ and $c$ are the things you need to know to write down the equation. We don't want them all to be zero as $0 = 0$ is not the equation of a line. We want $a$, $b$ and $c$ not all zero.

If we want the line to go through $P$ and $Q$, we get two equations to be satisfied by $a$, $b$ and $c$:

$$\begin{aligned} ap_1 + bp_2 - c &= 0 \\ aq_1 + bq_2 - c &= 0 \end{aligned}$$

or

$$\begin{aligned} p_1 a + p_2 b - c &= 0 \\ q_1 a + q_2 b - c &= 0 \end{aligned}$$

Since we have 2 equations for the 3 unknowns $a$, $b$ and $c$ (the points $P$ and $Q$ are supposed to be ones we know, and so $p_1, p_2, q_1$ and $q_2$ are all things we know) there are either going to be infinitely many solutions or none. We don't have none since $a = b = c = 0$ is a solution. So the equations are consistent. If we were to solve the equations by using (say) Gauss-Jordan elimination on

$$\left[\begin{array}{cccccc} p_1 & p_2 & -1 & : & 0 \\ q_1 & q_2 & -1 & : & 0 \end{array}\right]$$

we would surely end up with free variables and so infinitely many solutions. Taking a nonzero value for the free variable, we would get a nonzero solution for $a$, $b$ and $c$, so an equation for our line.

If we were to add in a third point $R = (r_1, r_2)$ we would get 3 equations rather than just 2

$$\begin{aligned} r_1 a + r_2 b - c &= 0 \\ p_1 a + p_2 b - c &= 0 \\ q_a 1 + q_2 b - c &= 0 \end{aligned}$$

and we don't expect more that one solution $a = b = c = 0$ when we have 3 equations in 3 unknowns.

If we think more carefully, we see that the only time we have other solutions is when there is actually a line through $R$, $P$ and $Q$, that is only when $R$ is on the line through $P$ and $Q$.

Using determinants we can then say what the condition is for the above 3 equations to have a solution other than $a = b = c = 0$. It is the same as requiring the matrix

$$\begin{bmatrix} r_1 & r_2 & -1 \\ p_1 & p_2 & -1 \\ q_1 & q_2 & -1 \end{bmatrix}$$

to have no inverse (see Theorem 5.13.2 (b)) and by Theorem 1.4.5 above, that is the same as

$$\det \begin{bmatrix} r_1 & r_2 & -1 \\ p_1 & p_2 & -1 \\ q_1 & q_2 & -1 \end{bmatrix} = 0$$

We are now going to treat $R$ as an unknown or variable point on the line through $P$ and $Q$. This determinant gives an equation that has to be satisfied by $R = (r_1, r_2)$. To emphasise the different rôle for the point $R$ (variable) and the points $P$ and $Q$ (fixed) we'll switch to writing $R = (x, y)$. Finally we get that

$$\det \begin{bmatrix} x & y & -1 \\ p_1 & p_2 & -1 \\ q_1 & q_2 & -1 \end{bmatrix} = 0$$

gives the equation of the line through $P = (p_1, p_2)$ and $Q = (q_1, q_2)$.

Multiplying the last column by $-1$ we could just as well write that equation as

$$\det \begin{bmatrix} x & y & 1 \\ p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \end{bmatrix} = 0$$

[We could have avoided the $-1$ if we wrote the equation of the line as $ax + by + c = 0$ (so not the same $c$) originally. Multiplying the column by $-1$ multiplies the determinant by $-1$ and minus the determinant equal zero is the same equation as without the minus.]

**1.12.1 Example.** Find the equation of the line through $P = (1, 2)$ and $Q = (4, 9)$ using determinants.

*Solution:* The equation is

$$\det \begin{bmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 4 & 9 & 1 \end{bmatrix} = 0$$

That works out as

$$x \det \begin{bmatrix} 2 & 1 \\ 9 & 1 \end{bmatrix} - y \det \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = 0$$

or

$$-7x + 3y + 1 = 0$$

**Circle through 3 points in the plane.** If we start with 3 points in the plane, there is usually one circle through the 3 points. There is a possibility we could pick 3 points in a line and then there would be no regular circle, rather a line, but we'll proceed without worrying about the case of collinear points.

First some words about the equations of circles. The circle centred at $(x_0, y_0)$ with radius $r > 0$ has equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

but if you multiply it out it is

$$x^2 + y^2 - 2x_0x - 2y_0y + (x_0^2 + y_0^2 - r^2) = 0.$$

Taking account of the fact that we can multiply the equation across by any nonzero number and still have the same equation, we'll take the equation to have the form

$$a(x^2 + y^2) + bx + cy + d = 0.$$

A genuine circle has $a \neq 0$ and there is another condition because the radius squared has to be positive, but we can think about that a little later. For now, recall that the unknown coefficients in the equation are the 4 numbers $a, b, c, d$ and they should not all be 0.

Now say we are given 3 points $P = (p_1, p_2)$, $Q = (q_1, q_2)$ and $R = (r_2, r_2)$ on the circle. This gives us 3 equations that the 4 unknowns $a, b, c, d$ have to satisfy:

$$\left\{ \begin{array}{rcl} a(p_1^2 + p_2^2) + bp_1 + cp_2 + d &=& 0 \\ a(q_1^2 + q_2^2) + bq_1 + cq_2 + d &=& 0 \\ a(r_1^2 + r_2^2) + br_1 + cr_2 + d &=& 0 \end{array} \right.$$

Since there are only 3 equations we have to get a nonzero solution. One approach would be to write out the equations as an augmented matrix like

$$\left[ \begin{array}{cccccc} p_1^2 + p_2^2 & p_1 & p_2 & 1 & : & 0 \\ q_1^2 + q_2^2 & q_1 & q_2 & 1 & : & 0 \\ r_1^2 + r_2^2 & r_1 & r_2 & 1 & : & 0 \end{array} \right]$$

and find a nonzero solution after using Gauss-Jordan elimination to find solution in terms of a free variable.

The method using determinants is to think of a fourth point $(x, y)$ and add in the condition that this point should be on the circle. Then we have 4 equations

$$\left\{ \begin{array}{ccccccccc} a(x^2 + y^2) & + & bx & + & cy & + & d & = & 0 \\ a(p_1^2 + p_2^2) & + & bp_1 & + & cp_2 & + & d & = & 0 \\ a(q_1^2 + q_2^2) & + & bq_1 & + & cq_2 & + & d & = & 0 \\ a(r_1^2 + r_2^2) & + & br_1 & + & cr_2 & + & d & = & 0 \end{array} \right.$$

Unless the point $(x, y)$ happens to be on the circle through $P$, $Q$ and $R$, there will be no $a, b, c$ and $d$ satisfying all 4 equations. Usually the matrix of coefficients

$$\left[ \begin{array}{cccc} x^2 + y^2 & x & y & 1 \\ p_1^2 + p_2^2 & p_1 & p_2 & 1 \\ q_1^2 + q_2^2 & q_1 & q_2 & 1 \\ r_1^2 + r_2^2 & r_1 & r_2 & 1 \end{array} \right]$$

will be invertible and that is the same as there being on the zero solution. Invertible means nonzero determinant.

But then, the equation

$$\det \begin{bmatrix} x^2+y^2 & x & y & 1 \\ p_1^2+p_2^2 & p_1 & p_2 & 1 \\ q_1^2+q_2^2 & q_1 & q_2 & 1 \\ r_1^2+r_2^2 & r_1 & r_2 & 1 \end{bmatrix} = 0$$

is exactly the condition for $(x, y)$ to be one of the points on the circle through $P$, $Q$ and $R$.

Expanding out the determinant we see why we have something that looks like the equation of a circle:

$$\begin{aligned} (x^2+y^2)\det \begin{bmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{bmatrix} - x\det \begin{bmatrix} p_1^2+p_2^2 & p_2 & 1 \\ q_1^2+q_2^2 & q_2 & 1 \\ r_1^2+r_2^2 & r_2 & 1 \end{bmatrix} & \\ +y\det \begin{bmatrix} p_1^2+p_2^2 & p_1 & 1 \\ q_1^2+q_2^2 & q_1 & 1 \\ r_1^2+r_2^2 & r_1 & 1 \end{bmatrix} - \det \begin{bmatrix} p_1^2+p_2^2 & p_1 & p_2 \\ q_1^2+q_2^2 & q_1 & q_2 \\ r_1^2+r_2^2 & r_1 & r_2 \end{bmatrix} &= 0 \end{aligned}$$

**Plane through 3 points in space.** The idea is again very similar, maybe marginally more complicated because it is in 3 dimensions.

Say we have 3 points $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ and $R = (r_1, r_2, r_3)$ and we want to know the equation of the plane in $\mathbb{R}^3$ that contains all 3 points. If the 3 points are collinear, there will be many such planes containing the line, but we will ignore this situation.

One thing to note is that we already know 2 ways to solve this. The most recent way used cross products (see Example 1.6.1). Now yet another way!

We write an equation for our plane

$$ax + by + cz = d$$

but it will be more convenient for us to have it

$$ax + by + cz - d = 0.$$

We should have $a, b, c$ not all zero for a real plane.

Using the 3 points $P$, $Q$, $R$ and also another unknown point $(x, y, z)$ on the plane we find 4 equations that $a, b, c$ and $d$ must satisfy

$$\begin{cases} ax + by + cz - d = 0 \\ ap_1 + bp_2 + cp_3 - d = 0 \\ aq_1 + bq_2 + cq_3 - d = 0 \\ ar_1 + br_2 + cr_3 - d = 0 \end{cases}$$

and then the condition for there to be a nonzero solution is

$$\det \begin{bmatrix} x & y & z & -1 \\ p_1 & p_2 & p_3 & -1 \\ q_1 & q_2 & q_3 & -1 \\ r_1 & r_2 & r_3 & -1 \end{bmatrix} = 0$$

This is the equation for the points $(x, y, z)$ on the plane.

Again we could make this simpler by multiplying column 4 by $-1$.

$$\det \begin{bmatrix} x & y & z & 1 \\ p_1 & p_2 & p_3 & 1 \\ q_1 & q_2 & q_3 & 1 \\ r_1 & r_2 & r_3 & 1 \end{bmatrix} = 0$$

**1.12.2 Example.** Find the equation of the plane that goes through $(1, 2, 3)$, $(3, 1, 2)$ and $(2, 3, 1)$.

*Solution:* Let $P = (1, 2, 3)$, $Q = (3, 1, 2)$ and $R = (2, 3, 1)$ and the above determinant equation to get

$$\det \begin{bmatrix} x & y & z & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} = 0$$

This expands to

$$x \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} - y \det \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} + z \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = 0$$

After some effort, this works out as

$$-3x - 3y - 3z + 18 = 0,$$

which we could maybe tidy up as $x + y + z = 6$. (Same answer as last time!)