1S11 (Timoney) Tutorial sheet 5
[October 23 – 26, 2012]

Name: Solutions

1. (a) Find parametric equations for the line in space that goes through both $(1, 2, 3)$ and $(-3, -2, 1)$.

   Solution: If $P = (1, 2, 3)$ and $Q = (-3, -2, 1)$ then $\vec{QP}$ is a vector parallel to the line. We can find $\vec{QP}$ by taking $\mathbf{P} - \mathbf{Q}$ where $\mathbf{P} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is the position vector of P and $\mathbf{Q} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is the position vector of Q. So

   $$\vec{QP} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

   and parametric equations are

   $$\begin{aligned} x &= 1 + 4t \\ y &= 2 + 4t \\ z &= 3 + 2t. \end{aligned}$$

   (We used P as one point on the line.)

   (b) Find cartesian equations for the same line.

   Solution: Recall that the cartesian equations take the form

   $$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3},$$

   where (x_0, y_0, z_0) are the coordinates of one point on the line (could be the constant terms in the parametric equations, the point for $t = 0$) and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is a vector parallel to the line (the coefficients of t in the parametric equations give us the components of this vector).

   So the answer in this case is

   $$\frac{x - 1}{4} = \frac{y - 2}{4} = \frac{z - 3}{2}.$$

   Aside. Another way to remember this is that solving each of the three parametric equations for t in terms of x, y and z gives

   $$t = \frac{x - 1}{4} = \frac{y - 2}{4} = \frac{z - 3}{2}.$$

2. Let $\mathbf{x} = (2, -1, 3, 5)$ and $\mathbf{y} = (3, 2, -4, -2)$ (in $\mathbb{R}^4$). Compute

   (a) $10\mathbf{x} - 7\mathbf{y}$

   Solution: $10\mathbf{x} - 7\mathbf{y} = (20, -10, 30, 50) + (-21, -14, 28, 14) = (-1, -24, 58, 64)$

(b) $\|\mathbf{x}\|$

Solution: $\|\mathbf{x}\| = \sqrt{2^2 + (-1)^2 + 3^2 + 5^2} = \sqrt{39}$.

(c) $\mathbf{x} \cdot \mathbf{y}$

Solution: $\mathbf{x} \cdot \mathbf{y} = (2)(3) + (-1)(2) + (3)(-4) + (5)(-2) = 6 - 2 - 12 - 10 = -18$

(d) the distance between $\mathbf{x}$ and $\mathbf{y}$.

Solution: $\sqrt{(2-3)^2 + (-1-2)^2 + (3+4)^2 + (5+2)^2} = \sqrt{1 + 9 + 49 + 49} = \sqrt{108} = 2\sqrt{27} = 6\sqrt{3}$

(e) the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$.

Solution: We know $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$ and so we know from earlier that $-18 = \sqrt{39}\|\mathbf{y}\| \cos\theta$. We still need to calculate $\|\mathbf{y}\| = \sqrt{3^2 + 2^2 + (-4)^2 + (-2)^2} = \sqrt{9 + 4 + 16 + 4} = \sqrt{33}$ and then we know

$$\cos\theta = \frac{-18}{\sqrt{39}\sqrt{33}} = \frac{-18}{3\sqrt{11}\sqrt{13}} = \frac{-6}{\sqrt{11}\sqrt{13}}$$

3. Find the equation of the hyperplane in $\mathbb{R}^4$ through $(1, 2, -6, 5)$ perpendicular to $(3, -2, 1, 8)$.

   Solution: The equation looks like $3x_1 - 2x_2 + x_3 + 8x_4 = \text{const}$ and $(1, 2, -6, 5)$ must satisfy the equation. So
   $$3(1) - 2(2) - 6 + 8(5) = \text{const}$$
   or $33 = \text{const}$. So the equation of the hyperplane is $3x_1 - 2x_2 + x_3 + 8x_4 = 33$.

Richard M. Timoney