Chapter 5. Infinite series

This material is in Chapter 9 of Anton Calculus.

Contents

5.1 Sequences

An *infinite sequence* (which we may often call just a *sequence*) if an infinite list (of numbers)

$$a_1, a_2, \ldots$$

Examples would be

$$1, 2, 3, 4, \ldots$$

where $a_n = n$, or

$$1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$$

where $a_n = 1/n^2$, or

$$1, -\frac{1}{2}, \frac{1}{3!}, -\frac{1}{4!}, \ldots$$

where a formula for the n^{th} number in the list is $a_n = (-1)^{n+1}/n!$. It is usual to refer to a_n as the n^{th} *term* of the sequence and sometimes to write that we are discussing a sequence

$$(a_n)_{n=1}^{\infty}$$

rather than listing $a_1, a_2, \ldots$. So we might say we want to consider the sequence

$$\left(\frac{1}{2^n}\right)_{n=1}^{\infty}$$

so that we have the formula for the n^{th} term given right away.

Our main concern will be limits of sequences (when they have limits).

5.1.1 Definition (Limit of a sequence)**.** If $(x_n)_{n=1}^{\infty}$ is a sequence (of numbers) $\lim_{n\to\infty} x_n = \ell$ means that x_n will certainly be close to ℓ for all large n.

More precisely we define it by saying that once we decide how close we mean when we say 'close to ℓ' there must be a suitable interpretation of largeness for n to make the previous assertion true.

If a sequence has a (finite) limit ℓ we say that it is a *convergent sequence*.

5.1.2 Remark. Thinking back over the various kinds of limits you have seen for limits of functions, such

$$\lim_{x\to a} f(x) = \ell, \qquad \lim_{x\to\infty} f(x) = \ell, \qquad \lim_{x\to a} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = \infty,$$

the definition for the limit of a sequence is most similar to the definition for $\lim_{x\to\infty} f(x) = \ell$.

5.1.3 Examples. (i) (Limits of constant sequences) If $x_n = k \in \mathbb{R}$ for all n, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} k = k$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Starting form these (fairly intuitive) examples, we can get quite far by using the next Theorem.

5.1.4 Theorem. *Suppose* $(x_n)_{n=1}^\infty$ *and* $(y_n)_{n=1}^\infty$ *are two convergent sequences with* $\lim_{n\to\infty} x_n = \ell_1$ *and* $\lim_{n\to\infty} y_n = \ell_2$.

(i) (Multiples) For $k \in \mathbb{R}$ *a fixed number,*

$$\lim_{n\to\infty} kx_n = k\ell_1$$

(ii) (Sums)

$$\lim_{n\to\infty} x_n + y_n = \ell_1 + \ell_2$$

(iii) (Products)

$$\lim_{n\to\infty} x_n y_n = \ell_1 \ell_2$$

(iv) (Quotients) Provided (y_n *is never 0 and)* $\ell_2 \neq 0$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\ell_1}{\ell_2}$$

This fact about limits of sums, products and quotients is proved in a way that mimics the proofs of the same results for limits of functions. We omit the proofs.

One thing to notice is that the limit of a sequence depends only on the terms for large n. A formal way to express that is that if you change just a finite number of terms of a sequence then you don't change the limit. Also if the sequence was not convergent, changing a finite number of terms will still give a sequence with no limit.

In the result about limits of quotients, the hypothesis that $\ell_2 \neq 0$ is somehow more important than the one about $y_n \neq 0$ for all n. The reason is that if $\ell_2 \neq 0$ then it must be that $y_n \neq 0$ for all n large enough (large enough that $|y_n - \ell_2|$ is smaller than $|\ell_2|$). So x_n/y_n will make sense for all large n if $\ell_2 \neq 0$. Of course then it is a problem that the sequnce of fractions $(x_n/y_n)_{n=1}^\infty$ might not make real sense if y_n is ever 0. But we could make some arbitrary convention about what to replace x_n/y_n by when $y_n = 0$ and then the result would hold as long as $\ell_2 \neq 0$.

5.1.5 Examples. (i) An example of a sequence with no limit is the alternating sequence

$$-1, 1, -1, 1, -1, \ldots$$

which we could describe more succinctly as $((-1)^n)_{n=1}^{\infty}$.

(ii) If x_n is given by a rational function of n (meaning $x_n = p(n)/q(n)$ where p and q are polynomials) where the degree of the numerator is no larger than the degree of the denominator, we can find $\lim_{n\to\infty} x_n$ by dividing above and below by the highest power of n in the denominator and using the theorem 5.1.4 and the examples 5.1.3 repeatedly.

For instance

$$\begin{aligned}\lim_{n\to\infty} \frac{4n^3 - n^2 + 11n + 102}{7n^3 + n - 3} &= \lim_{n\to\infty} \frac{4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{7 + \frac{1}{n^2} - 3\frac{1}{n^3}} \\ &= \frac{\lim_{n\to\infty} 4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{\lim_{n\to\infty} 7 + \frac{1}{n^2} - 3\frac{1}{n^3}}\end{aligned}$$

using Theorem 5.1.4 (iv) — which will be justified once we can show that the limits in the numerator and denominator do make sense. Looking first at the numerator, we know (by 5.1.3) $\lim_{n\to\infty} 4 = 4$ and $\lim_{n\to\infty} 1/n = 0$, hence $\lim_{n\to\infty} -11/n = 0$ (by 5.1.4 (i) and so $\lim_{n\to\infty}(4 - 11/n) = 4$ (by 5.1.4 (ii)). Next $\lim_{n\to\infty} 1/n^2 = \lim_{n\to\infty}(1/n)(1/n) = 0$ (by 5.1.4 (iii)) and by the same result on limits of products $\lim_{n\to\infty} 1/n^3 = \lim_{n\to\infty}(1/n^2)(1/n) = 0$. Using 5.1.4 (i) and 5.1.4 (ii) again, we deduce

$$\lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2}\right) = \lim_{n\to\infty}\left(4 - 1\frac{1}{n}\right) + \lim_{n\to\infty} 11\frac{1}{n^2} = 4 + 11(0) = 4$$

and finally we have the limit of the numerator

$$\lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2} + 120\frac{1}{n^3}\right) = \lim_{n\to\infty}\left(4 - 1\frac{1}{n} + 11\frac{1}{n^2}\right) + \lim_{n\to\infty} 120\frac{1}{n^3} = 4 + 120(0) = 4$$

Similarly, but compressing the various steps, we have the limit of the denominator

$$\lim_{n\to\infty}\left(7 + \frac{1}{n^2} - 3\frac{1}{n^3}\right) = 7 + 0 - 3(0) = 7$$

and the limit of the original sequence is then justfied as a limit of a quotient

$$\lim_{n\to\infty} \frac{4n^3 - n^2 + 11n + 102}{7n^3 + n - 3} = \frac{\lim_{n\to\infty} 4 - \frac{1}{n} + 11\frac{1}{n^2} + 102\frac{1}{n^3}}{\lim_{n\to\infty} 7 + \frac{1}{n^2} - 3\frac{1}{n^3}} = \frac{4}{7}$$

5.1.6 Theorem. *(useful theorem)*

(i) If $(x_n)_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty} x_n = \ell$ and $f(x)$ makes sense on both sides of ℓ and is continuous at ℓ, then

$$\lim_{n\to\infty} f(x_n) = f(\ell)$$

(ii) *If* $\lim_{x\to\infty} f(x) = \ell_f$ *and now* $(x_n)_{n=1}^\infty$ *is a sequence with* $\lim_{n\to\infty} x_n = \infty$, *then*

$$\lim_{n\to\infty} f(x_n) = \ell_f$$

(Strictly we should have defined what $\lim_{n\to\infty} x_n = \infty$ *means — it means that* x_n *is large for all large* n. *Or, more precisely, if you pick* M, *then there is* N *so that*

$$x_n > M \text{ holds as long as } n > N.$$

So whatever M *you decide, or whatever you mean by "*x_n *is large", then there is a choice of* N *or "large* n*" to make it true that "*x_n *is large for all large* n*".)*

5.1.7 Example. (i) $\lim_{n\to\infty} 1/\sqrt{n} = 0$ because $\lim_{x\to 0} \sqrt{|x|} = 0$ and $\lim_{n\to\infty} 1/n = 0$ (by Theorem 5.1.6 (i)). We are using $1/\sqrt{n} = \sqrt{1/n}$.

(ii) Since $\cos x$ is continuous everywhere

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{2} - \frac{1}{n}\right) = \cos\frac{\pi}{2} = 0$$

(by Theorem 5.1.6 (i)).

(iii) It should not be a surprise that $\lim_{n\to\infty} n = \infty$. We know from our discussion of the exponential function that

$$\lim_{x\to\infty} \frac{x}{e^x} = 0$$

(in fact that e^x grows faster than any power of x). So, from Theorem 5.1.6 (ii)),

$$\lim_{n\to\infty} \frac{n}{e^n} = 0$$

(iv) If we put $x = \ln t$ in $\lim_{x\to\infty} \frac{x}{e^x} = 0$ we get

$$\lim_{t\to\infty} \frac{\ln t}{t} = 0$$

(You might be able re member this as saying that $\ln t$ grows [much] more slowly that t as $t \to \infty$.)

Using the theorem, we get

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

(v)

$$\lim_{n\to\infty} \frac{\cos n \ln n}{n} = 0$$

follows because $|\cos n| \leq 1$ and so

$$\frac{\cos n \ln n}{n}$$

is closer to 0 than

$$\frac{\ln n}{n}$$

5.2 Monotone sequences

5.2.1 Definition. A sequence $(x_n)_{n=1}^\infty$ (in $\mathbb{R}$) is called *monotone increasing* if $x_n \leq x_{n+1}$ for each n.

So that means $x_1 \leq x_2 \leq x_3 \leq \cdots$ and it implies that $x_n \leq x_m$ if $n < m$.

A sequence $(x_n)_{n=1}^\infty$ is called *monotone decreasing* if $x_n \geq x_{n+1}$ for each n.

A sequence is called *monotone* is it is either monotone increasing *or* monotone decreasing.

A sequence $(x_n)_{n=1}^\infty$ is called *strictly monotone decreasing* if $x_n > x_{n+1}$ for each n.

A sequence $(x_n)_{n=1}^\infty$ is called *strictly monotone increasing* if $x_n < x_{n+1}$ for each n.

A sequence is called *strictly monotone* is it is either strictly monotone increasing *or* strictly monotone decreasing.

5.2.2 Examples. (i) If $x_n = 1/n$, then $(x_n)_{n=1}^\infty = \left(\frac{1}{n}\right)_{n=1}^\infty$ is a monotone decreasing sequence, in fact a strictly monotone decreasing one.

(ii) On the other hand $x_n = 1 - \frac{1}{n^2}$ is a (strictly) monotone increasing sequence which has $\lim_{n\to\infty} x_n = 1 =$ the smallest number $\geq x_n$ for each n.

(iii) If $(x_n)_{n=1}^\infty$ is any monotone increasing sequence, then $(-x_n)_{n=1}^\infty$ is monotone decreasing.

Using this one can prove things about monotone decreasing sequences once one has proved the similar fact about increasing sequences.

(iv) The sequence $((-1)^n)_{n=1}^\infty$ is not monotone.

The following is actually quite a big theorem if we wanted to prove it, but it seems reasonably obvious if you don't want to prove it (and we will not).

5.2.3 Theorem. *If $(x_n)_{n=1}^\infty$ is any monotone increasing sequence then* either

(i) there is a number M (called an uper bound*) so that $x_n \leq M$ for each n and in that case*

$$\lim_{n\to\infty} x_n \textit{ exists as a finite value}$$

(and it is actually the least possible upper bound, the least M that works)

or

(ii) there is no such upper bound, and in that case

$$\lim_{n\to\infty} x_n = \infty$$

5.2.4 Corollary. *If $(x_n)_{n=1}^\infty$ is any monotone decreasing sequence with a lower bound in $\mathbb{R}$ (i.e. a number L so that $x_n \geq L$ holds for all n), then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.*

5.2.5 Remark. Monotone sequences are then somewhat simpler that general sequences, at least in terms of whether they have limits or not. If they are bounded they do have (finite) limits.

The term 'convergent sequence' is usually reserved for those with finite limits.

5.3 Series

A *series* (of real numbers) is in fact just a sequence $(x_n)_{n=1}^\infty$ but we use the word 'series' when we are thinking of adding up the infinite list of numbers, to try and make sense of

$$x_1 + x_2 + x_3 + \cdots$$

Such an infinite sum does not make sense without a definition of what it might mean. There is no theoretical difficulty with any finite sum (though there could be a practical difficulty if the number of terms is really huge).

5.3.1 Notation. We write $\sum_{n=1}^\infty x_n$ to indicate a sequence $(x_n)_{n=1}^\infty$ considered as a series.

The *partial sums* of a series $\sum_{n=1}^\infty x_n$ are the finite sums

$$s_n = x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j$$

5.3.2 Definition. A series $\sum_{n=1}^\infty x_n$ is called *convergent* if the sequence of its partial sums has a finite limit, that is if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{j=1}^n x_j$ exists in $\mathbb{R}$.

If the series $\sum_{n=1}^\infty x_n$ is convergent then the limit of the partial sums, $\lim_{n\to\infty} \sum_{j=1}^n x_j$ is defined to be the *sum* of the series.

In a notation that is totally standard but also potentially confusing, $\sum_{n=1}^\infty x_n$ is used for two things. One is to say that we are discussing a series, wondering then whether or not it converges, and the other is to represent the sum of the series when it does converge:

$$\sum_{n=1}^\infty x_n = \lim_{n\to\infty} \sum_{j=1}^n x_j$$

Another term used for convergent series is *summable series*.

5.3.3 Proposition. *(Link to monotone sequences)*

If $\sum_{n=1}^\infty x_n$ is a series of nonnegative terms then its sequence of partial sums is monotone increasing.

Proof. As usual we will write $s_n = \sum_{j=1}^n x_j = x_1 + x_2 + \cdots + x_n$ for the n^{th} partial sum.

We can then observe that $s_{n+1} - s_n = x_{n+1} \geq 0$ and so $s_n \leq s_{n+1}$ always.

This says that $(s_n)_{n=1}^\infty$ is monotone increasing. □

5.3.4 Theorem. *If $\sum_{n=1}^\infty x_n$ is a series of nonnegative terms then it is convergent if and only if its sequence of partial sums is bounded above.*

(We will skip a detailed proof, but it follows from Theorem 5.2.3 applied to the sequence of partail sums.)

5.3.5 Examples. There are a small number of series where we can show that they are convergent and also compute their sums by direct and elementary methods.

(i) (Telescoping sums)

Suppose the series $\sum_{n=1}^{\infty} x_n$ has n^{th} term

$$x_n = \frac{1}{n} - \frac{1}{n+1}$$

which could also be rewritten

$$x_n = \frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n^2+n}$$

Then the partial sums are

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

So the sum of the series is

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$$

(ii) (not such a good example) If $x_n = 1$ for each n, then $s_n = \sum_{j=1}^{n} 1 = n$ and so $\lim_{n\to\infty} s_n$ is infinite. The series

$$\sum_{n=1}^{\infty} 1$$

does *not* converge.

(iii) (Geometric series)

Suppose there there is a number r with $|r| < 1$ and $x_n = r^{n-1}$. The sequence $1, r, r^2, \ldots$ is called a *Geometric Progression* (or GP) and the corresponding series is called a *Geometric series*. At least if $r \neq 0$ the ratio of successive terms is

$$\frac{x_{n+1}}{x_n} = \frac{r^{n+1}}{r^n} = r$$

(constant ratio r, also known as a common ratio).

There is a formula for the partial sums

$$s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$$

There are various ways to prove this, for instance by induction on n, but a simple proof is to write down

$$rs_n = r + r^2 + r^3 + \cdots + r^n$$

and subtract that from s_n (cancelling all the common terms in the sums) to get

$$s_n - rs_n = 1 - r^n$$

That gives $(1-r)s_n = 1 - r^n$ and then we can divide by $1-r$ to get the formula. [By the way that formula is good for $r \neq 1$, not just for $|r| < 1$.]

Now

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} r^{n-1} = \lim_{n\to\infty} \sum_{j=1}^{n} r^{j-1} = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}$$

Here we are relying on $\lim_{n\to\infty} r^n = 0$ for $|r| < 1$, which seems like it must be true if you look at it (but we will not prove carefully).

The main point of the next few results is to get an understanding of which series are convergent (or summable). However the short answer is that it is a rather subtle question, with no simple method to decide about convergence — at least no method that always works.

5.3.6 Theorem. *(Terms must tend to 0)*
If $\sum_{n=1}^{\infty} x_n$ is a convergent series, then

$$\lim_{n\to\infty} x_n = 0$$

Proof. Write s_n as usual for the n^{th} partial sum $s_n = \sum_{j=1}^{n} x_j$ and $s = \sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$.
Notice that

$$s_{n+1} - s_n = (x_1 + x_2 + \cdots + x_{n+1}) - (x_1 + x_2 + \cdots + x_n) = x_{n+1}$$

Therefore we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = s - s = 0$$

It follows that $\lim_{n\to\infty} x_n = 0$ (x_n small for large n). □

Notice that it is vital that $s \in \mathbb{R}$ (a finite sum). If we allowed series to have sum ∞ (or $-\infty$ either) then we could not make sense of $s - s$.

5.3.7 Theorem (Integral test)**.** *Suppose $f\colon [1,\infty) \to [0,\infty)$ is a decreasing continuous function. Then the series*

$$\sum_{n=1}^{\infty} f(n) \text{ converges}$$

if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^{b} f(x)\,dx$$

converges (to a finite value).

(Note that since $f(x) \geq 0$ the integrals $\int_1^b f(x)\,dx$ increase when b increases. If these numbers are bounded above, that is if all the integrals are $\leq M$ for some fixed number M, then their limit exists and is the smalled such upper bound M.)

Proof. The proof is based on the fact that

$$f(n) \geq f(x) \geq f(n+1)$$

for $n \leq x \leq n+1$ (because $f(x)$ is decreasing with x) and so

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

which means

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

It follows that

$$\sum_{j=1}^n f(j) \geq \sum_{j=1}^n \int_j^{j+1} f(x)\,dx \geq \sum_{j=1}^n f(j+1)$$

and so

$$\sum_{j=1}^n f(j) \geq \int_1^{n+1} f(x)\,dx \geq \sum_{j=2}^{n+1} f(j) = \left(\sum_{j=1}^{n+1} f(j)\right) - f(1)$$

So if the series $\sum_{n=1}^\infty f(n)$ converges, there is a finite upper bound s for its partial sums and we have

$$\int_1^{n+1} f(x)\,dx \leq \sum_{j=1}^n f(j) \leq s$$

for all n. For any $b \geq 1$ we can find n with $b \leq n+1$ and then

$$\int_1^b f(x)\,dx \leq \int_1^{n+1} f(x)\,dx \leq \sum_{j=1}^n f(j) \leq s$$

So the integrals must be bounded above if the series converges.

On the other hand if there is $M \in \mathbb{R}$ with $\int_1^b f(x)\,dx \leq M$ for all $b \geq 1$, then we have

$$\sum_{j=1}^{n+1} f(j) \leq f(1) + \int_1^{n+1} f(x)\,dx \leq f(1) + M$$

The partial sums of the series $\sum_{n=1}^\infty f(n)$ are bounded above (by $f(1) + M$) and so the series converges (Theorem 5.3.4). □

The point of the next example is that Theorem 5.3.6 tells us a property that every convergent series has, but it does not work in reverse. If $\lim_{n\to\infty} x_n = 0$ it may still be the case that the series $\sum_{n=1}^\infty x_n$ fails to be summable.

5.3.8 Example. The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is *not convergent*,

(This is a famous example and has a special name — it is called the *harmonic series*.)

Proof. We already know that $\int_1^\infty (1/x)\,dx$ does not converge.

By the integral test (Theorem 5.3.7) $\sum_{n=1}^\infty \frac{1}{n}$ does not converge either. □

Here is another straightforward use of the integral test.

5.3.9 Example. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Proof. We saw before in Chapter 3 that

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges.

By the integral test (Theorem 5.3.7) $\sum_{n=1}^\infty \frac{1}{n}^2$ converges. □

5.3.10 Theorem (Comparison test for series of positive terms)**.** *Suppose $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty y_n$ are two series with nonnegative terms ($x_n \geq 0$ and $y_n \geq 0$) and suppose $x_n \leq y_n$ for each n. Then*

(a) if the larger series $\sum_{n=1}^\infty y_n$ converges then so does the smaller series $\sum_{n=1}^\infty x_n$;

(b) if the smaller series $\sum_{n=1}^\infty x_n$ does not converge then neither does the larger one $\sum_{n=1}^\infty y_n$.

Proof. Notice that the partial sums of the smaller series are smaller than those of the larger: $\sum_{j=1}^n x_j \leq \sum_{j=1}^n y_j$ for each n. Using Theorem 5.3.4, if $\sum_{n=1}^\infty y_n$ converges then there is $u \in \mathbb{R}$ so that $\sum_{j=1}^n y_j \leq u$ for all n. But then $\sum_{j=1}^n x_j \leq u$ also and so $\sum_{n=1}^\infty x_n$ converges (using Theorem 5.3.4). That shows (a). And (b) follows from (a). □

5.3.11 Proposition. *If $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty y_n$ are two convergent series, then $\sum_{n=1}^\infty (x_n + y_n)$ is also convergent and has sum*

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$$

We skip the proof, but is is rather easy.

5.3.12 Remark. An equally simple fact is that if $\sum_{n=1}^{\infty} x_n$ converges and k is a constant, then

$$\sum_{n=1}^{\infty}(kx_n) = k\sum_{n=1}^{\infty} x_n$$

5.3.13 Definition. A series $\sum_{n=1}^{\infty} x_n$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent.

5.3.14 Theorem. *Absolutely convergent series are convergent.*
That is if $\sum_{n=1}^{\infty} |x_n|$ converges, then so does $\sum_{n=1}^{\infty} x_n$.

This can be proved using the comparison test, put perhaps the proof is a little tricky and so we will omit it.

5.3.15 Example. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges.

Proof. Since $|\cos x| \leq 1$ always,

$$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

Since we know $\sum_{n=1}^{\infty} 1/n^2$ converges, the comparison test tell us that $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ converges.
So now we know $\sum_{n=1}^{\infty}(\cos n)/n^2$ is absolutely convergent, therefore convergent. □

5.3.16 Theorem (Ratio test)**.** *Suppose $\sum_{n=1}^{\infty} x_n$ is a series and*

$$r = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists (as a finite nonnegative limit or as ∞).

(i) if $r < 1$ then the series is (absolutely) convergent;

(ii) if $r > 1$ then the series does not converge;

(iii) if $r = 1$ no conclusion can be drawn about convergence of the series from this information alone.

Note that the limit we are considering is the limit of the ratio of a term of the series to the previous one. To be more precise the absolute value of that ratio.

The proof of the ratio test ($R < 1$ case) relies on comparing the series $\sum_{n=1}^{\infty} |x_n|$ to a geometric series $\sum_{n=1}^{\infty} M\rho^n$ for some M and ρ with $r < \rho < 1$. Again we will omit the proof.

5.3.17 Example. For which values of x does $\sum_{n=1}^{\infty} \sqrt{n}x^n$ converge?
Solution: Although x is a variable, for fixed x we have a series $\sum_{n=1}^{\infty} t_n$ where the n^{th}-term is

$$t_n = \sqrt{n}x^n$$

To apply the ratio test, we should evaluate

$$r = \lim_{n\to\infty} \frac{|t_{n+1}|}{|t_n|} = \lim_{n\to\infty} \frac{|\sqrt{n+1}x^{n+1}|}{|\sqrt{n}x^n|} = \lim_{n\to\infty} \sqrt{1+1/n}|x| = |x|$$

Since there is a limit we can say that the series definitely converges for $|x| < 1$ (that is for $-1 < x < 1$) and does not converge for $|x| > 1$. The uncertain values of x are just $x = \pm 1$.

In many problems of this type, it can be difficult to deal with these remaining cases. However, in this example the series for $x = 1$ is $\sum_{n=1}^{\infty} \sqrt{n}$ and for $x = -1$ it is $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$. Neither of these can converge because the terms do not tend to 0 as $n \to \infty$.

5.4 Power series

5.4.1 Definition. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where x is a variable, a is a fixed point called the centre of the series and the numbers a_0, a_1, ... are called the coefficients of the series.

In series of this type it is usual to have a convention that $(x-a)^0$ means 1, even for $x = a$ (where it would read 0^0, something that one should not normally attempt to define).

So the power series means

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

When $a = 0$ the series looks like

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

5.4.2 Example. In the previous example we have $a = 0$ and $a_n = \sqrt{n}$.

We don't have any simple formula for the sum of that series. From the formula for the sum of a geometric series we know

$$1 + (x-a) + (x-a)^2 + \cdots = \sum_{n=0}^{\infty} (x-a)^n = \sum_{n=1}^{\infty} (x-a)^{n-1} = \frac{1}{1-(x-a)}$$

for $|x - a| < 1$. (Here the coefficient of $(x-a)^n$ is $a_n = 1$ for each $n \geq 0$.)

We now summarise (without any attempt at a proof) some of the main facts about power series.

5.4.3 Theorem. *Let $\sum_{n=1}^{\infty} a_n (x-a)^n$ be a power series.*

(a) *The series has a* radius of convergence*, a 'number'* $R \geq 0$ *with the property that the series converges for* $|x-a| < R$ *but fails to converge for any* x *with* $|x-a| > R$*.*

However the 'number' R *may be* ∞*, that is it may be that the power series converges for all* x*. (It may also be* $R = 0$*, in which case the series converges only for* $x = a$*.)*

If $R > 0$ *we can define* $f\colon (a-R, a+R) \to \mathbb{R}$ *by*

$$f(x) = \sum_{n=1}^{\infty} a_n (x-a)^n$$

The function $f(x)$ *will be diffentiable and its derivative is given by*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

for $|x-a| < R$*.*

(b) *Moreover,if* $R > 0$*, the series* $\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ *has the same radius of convergence and so* $f'(x)$ *is differentiable.*

(c) *In fact, assuming* $R > 0$*,* $f(x)$ *can be differentiated any number of times and the coefficients* a_n *are related to the values of the derivatives at* a *by*

$$a_n = \frac{1}{n!} f^{(n)}(a).$$

(d) *We can also integrate 'term by term' rather than differentiate and say that*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$$

has the same radius of convergence R *and is an antiderivative for* $f(x)$ *(when* $|x-a| < R$*).*

5.4.4 Example. The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

has radius of convergence $R = \infty$.

To prove that we can use the ratio test to show that the series converges for each x.

For the ratio test we take the limit of the absolute value of the ratios of each term to the term before

$$\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)!} x^{n+1}\right|}{\left|\frac{1}{n!} x^n\right|} = \lim_{n\to\infty} \frac{n! |x|^{n+1}}{(n+1)! |x|^n} = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0$$

Therefore, by the ratio test, the series converges for all x and that tells us $R = \infty$ is the radius of convergence.

It follows then that we can define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

and then (according to Theorem 5.4.3) we can calculate $f'(x)$ by differentiating the series 'term by term'. We get

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \cdots = f(x)$$

Since $f(0) = 1$, we can say that $f(x)$ is e^x.

This would justify the existence of f with $f' = f$ and $f(0) = 1$.

5.4.5 Example. We know the sum of a geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

for $|x| < 1$, while the series does not converge for $|x| > 1$. [Why not? Well the terms x^n do not tend to 0 as $n \to \infty$ and so there is no chance that the series can converge if $|x| > 1$.]

So this a a power series (centered at $a = 0$) with radius of convergence $R = 1$. For $|x| < 1$ we must have the derivative

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^n$$

or

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

Instead of differentiating we could also integrate the series. We get

$$\frac{d}{dx}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

(for $|x| < 1$) and so

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right)$$

In this way it is possible to arrive at formulae for sums of series. For example take $x = 1/2$ in the last formula. We get $\sum_{n=1}^{\infty} 1/(n2^n) = \ln 2$.

5.4.6 Remark. If we knew about the exponential function $f(x) = e^x$ and we were hoping to find a series (centered at $a = 0$ say) that summed to it, we know we would have to take the coefficients to be

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{e^a}{n!}$$

(Look at Theorem 5.4.3 (c)). In the case of the exponential function, it turns out that $\sum_{n=0}^{\infty} a_n(x-a)^n = e^x$ works for all x, but that does not happen for every function. Sometimes it works, sometimes for all x, sometimes with a finite radius of convergence and sometimes it just does not work except for $x = a$.

There is a theorem called Taylor's theorem which can sometimes be used to prove that $f(x)$ is given by a power series.

5.4.7 Definition. If $f(x)$ is n-times differentiable at a, then the *Taylor polynomial of degree n for f centered at a* is

$$P_{n,f,a}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j = f(a)+f'(a)(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Note that they would be the partial sums of the power series, if there was a power series that gave $f(x)$ near $x = a$.)

Taylor's Theorem gives a formula for $f(x) - P_{n,f,a}(x)$ (which sometimes can be used to show $\lim_{n\to\infty} P_{n,f,a}(x) = f(x)$). We won't explain that in any more detail.

5.4.8 Remark. For $n = 1$, the Taylor polynomial of degree 1 is

$$P_{1,f,a}(x) = f(a) + f'(a)(x - a)$$

and the linear approximation formula says that (provided $f'(a)$ exists)

$$f(x) \cong f(a) + f'(a)(x - a)$$

is good approximation for $x - a$ small. The idea of Taylor's theorem is that the quadratic approximation

$$f(x) \cong f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

should be even better (again when $x - a$ is small).

Taylor's theorem gives a way to analyse the error in an approximation of this kind.