Chapter 4. Differential equations

This material is in Chapter 8 of Anton Calculus.

Contents

4.1 What are differential equations?

Answer: equations where the unknown is a function $y = y(x)$ and the equation involves derivatives of y.

If the equation involves just the first derivative dy/dx (and possibly x and y as well), we call it a *first order* differential equation. The equation $dy/dx = y$ was used to introduce the exponential function (in our §2.1).

If the equation involves the second derivative d^2y/dx^2 (and possibly x, y and dy/dx as well), we call it a *second order* differential equation. We could also discuss higher order equations.

A (first order) differential equation like

$$\frac{dy}{dx} = x \cos x$$

(derivative of y equal some function of x) is really just an indefinite integral

$$y = \int x \cos x \, dx$$

and these are considered "easy" or regarded as "solved" differential equations (via techniques of integration). As in the case of indefinite integrals, the solution of a first order differential equation will typically have one arbitrary constant in it. Some side condition (like the value of y at $x = 0$) needs to be given to work out the constant.

Second order differential equations will usually have two arbitrary constants in their solutions (and then two side conditions will be needed to work out both constants).

A simple example (which would be counted as a trivial case of a second order differential equation) is

$$\frac{d^2y}{dx^2} = x^2 + x$$

We can see that we must then have

$$\frac{dy}{dx} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C_1$$

(where C_1 is some constant). Then we must have

$$y = \frac{1}{12}x^4 + \frac{1}{6}x^3 + C_1 x + C_2$$

for some (other) constant C_2.

4.2 Where do differential equations arise?

The answer is basically that many practical problems turn into differential equations when turned into a mathematical problem. Some examples were already given in Example 2.1 concerning exponential growth (or decay).

In mechanics, Newton's second law (mass times acceleration equals force) translates immediately into a second order differential equation because acceleration is the second derivative of position. Sometimes force can depend on velocity (the first derivative of position), for example if there is friction, or force can depend on position (for example a rocket will experience a gravitational force depending on its position).

Realistically things often move in space and then these differential equations are vector equations, or systems of differential equations with the unknown functions being the components of the position vector as functions of time. These would then be systems of (say 3) differential equations. A simple case where Newton's laws can be realistic with one dimensional motion is the oscillation (or vibration) of a spring that is constrained to move along the axis of the spring (just stretch and compress with no sideways motion). Sometimes this is referred to as a simple harmonic oscillator and the differential equation is of the type

$$\frac{d^2x}{dt^2} = -kx$$

where $k > 0$ is a constant related to the strength of the spring (and $x = 0$ is the case where the spring is neither stretched nor compressed). This differential equation is more complicated than the ones we will study but the general solution is actually known to be

$$x(t) = C_1 \cos \sqrt{k}t + C_2 \sin \sqrt{k}t$$

(and are periodic with period $2\pi/\sqrt{k}$).

Differential equations also arise in describing current flow in electrical networks (specifically in LRC circuits). In fact many physical problems become differential equations when modelled using mathematics, but there are also cases where the functions to be found are functions of more than one variable and then they are often translated into *partial differential equations*. If $z = f(x, y)$ is a function of two variables, then the partial derivative of $f(x, y)$ with respect to x is found by differentiating with respect to x while holding y constant. It can be denoted $\frac{\partial z}{\partial x}$. Similarly, the partial derivative $\frac{\partial z}{\partial y}$ is got by treating x as a constant and differentiating with respect to y. An example of a situation where a partial differential equation could arise would be where $y = u(x,t)$ describes the shape of a violin string at time t (over $0 \leq x \leq L$, with L the length between the points supporting the string) and where we assume the string vibrates up and down only ($y = u(x,t)$ is the vertical displacement at time t of the part of the string originally at position x units along; or if you freeze time t then the graph of $y = u(x,t)$ as a function of x gives a snapshot of the shape of the string at that time). Then u satisfies a *wave equation* like

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(where c is a constant depending on the characteristics of the string). Since we have not learned about partial derivatives yet, or even anything about functions that depend on more than one variable, we are certainly not in a position to discuss partial differential equations (PDEs).

We will limit ourselves to a few very easy kinds of (first order) ordinary differential equations (sometimes referred to as ODEs).

4.3 Variables separable

(Anton §8.2)

These are equations that are only a little more complicated than indefinite integrals, ones that can be written with a function of the unknown y times the derivative of y equals a function of x, that is of the form

$$h(y)\frac{dy}{dx} = g(x)$$

or sometimes this is written

$$h(y)\,dy = g(x)\,dx$$

Here is an example

$$\frac{dy}{dx} = x^2 y$$

It is not exactly of the form stated but (as long as $y \neq 0$) we can write it as

$$\frac{1}{y}\frac{dy}{dx} = x^2$$

or

$$\frac{1}{y}\,dy = x^2\,dx$$

Then we can integrate both sides and get

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

or

$$\ln|y| + C_1 = \frac{1}{3}x^3 + C_2$$

We can amalgamate the two constants C_1 and C_2 into one $C = C_2 - C_1$ and then

$$\ln|y| = \frac{1}{3}x^3 + C$$

Now we have an equation that is no longer a differential equation (an equation that at least implicitly gives y in terms of x) and from the point of view of differential equation theory, it is now "solved". In this case we can solve it more explicitly and get

$$|y| = e^{x^3/c+C}$$

We need a further piece of information about the solution to find y, such as (say) $y = 1$ at $x = 1$. Then

$$1 = e^0 = e^{1/3+C}$$

and so we need $C = -1/3$ and $y > 0$. We then have

$$y = e^{x^3/c-1/3} = e^{(x^3-1)/3},$$

a complete solution.

The original equation did also allow the (constant) solution $y \equiv 0$ (but that won't fit with the condition $y = 1$ at $x = 1$).

4.4 Linear constant coefficient first order

(Anton §8.4)

These are the equations like those already discussed in Example 2.1, of the form

$$\frac{dy}{dx} - ky = g(x)$$

They can be solved by the 'integrating factor trick' of multiplying across by e^{-kx} so that the equation becomes

$$\frac{d}{dx}(e^{-kx}y) = g(x)e^{-kx}.$$

This can be solved by integrating both sides (with respect to x) to get

$$e^{-kx}y = \int g(x)e^{-kx}\,dx.$$

When you are done with the integral multiply across by e^{kx} to find y.

We will end up with an arbitrary constant and we need another bit of information about the solution to find the constant.

4.4.1 Example. Solve

$$\frac{dy}{dx} - 3y = x$$

subject to $y = 0$ at $x = 0$.
Solution: The integration factor is e^{-3x} and multiplying by it gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = xe^{-3x}$$

and this is the same as (by the product rule)

$$\frac{d}{dx}\left(e^{-3x}y\right) = xe^{-3x}$$

So

$$e^{-3x}y = \int xe^{-3x}\,dx$$

and we can integrate that by parts with $u = x$, $dv = e^{-3x}\,dx$, $du = dx$, $v = -(1/3)e^{-3x}$. That gives

$$e^{-3x}y = \int u\,dv = uv - \int v\,du = -\frac{x}{3}e^{-3x} + \int \frac{1}{3}e^{-3x}\,dx = -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + C$$

Multiply by e^{3x} to get

$$y = -\frac{x}{3} - \frac{1}{9} + Ce^{3x}$$

For $y = 0$ at $x = 0$ we need $0 = 0 - (1/9) + C$ or $C = 1/9$. So, finally,

$$y = -\frac{x}{3} - \frac{1}{9} + \frac{1}{9}e^{3x}$$

4.4.2 Remark. Note that in the book this integrating factor method is extended to the case when k is not a constant, but a function of x, in which case the integrating factor is more complicated, We won't pursue differential equations in any more detail in this module.