Chapter 3. Techniques of integration

This material is in Chapter 7 of Anton Calculus.

Contents

3.1 Recap: Integration in one variable

There are two topics with similar names:

- Reverse of differentiation

  Indefinite integral

  $$\int f(x)\,dx = \text{ most general antiderivative for } f(x)$$

  The term 'antiderivative for $f(x)$' means a function whose derivative turns out to be $f(x)$.

- Definite integral

  This is related to summation (it is a limit of sums of a certain kind). The integral sign $\int$ was originally invented as a modified S (for sum).

There is no reason to expect a connection between these two different things, but there is.

3.1.1 Theorem (Fundamental Theorem of Integral Calculus)**.** *Assume that $y = f(x)$ is continuous for $a \leq x \leq b$. Consider $A(x) = \int_a^x f(t)\,dt$ for $a \leq x \leq b$. ($A(x)$ is a new function, built from f and definite integration.) Then $A(x)$ is an antiderivative for f (that is $A'(x) = f(x)$ for $a \leq x \leq b$).*

In summary:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \qquad (a \leq x \leq b, \text{ if } f \text{ continuous})$$

This is one part of the Fundamental theorem, or one way to state it.

3.1.2 Corollary. *There is an antiderivative for every continuous function f.*

3.1.3 Example. Find $\displaystyle\frac{d}{dx}\left(\int_a^x \frac{1}{t}\,dt\right)$
Solution: By the theorem

$$\frac{d}{dx}\left(\int_a^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

Well, what about the hypothesis that the integrand, $f(t) = 1/t$ in this example should be continuous? For that we need to avoid $t = 0$. We need to suppose that 0 is not in the closed interval from a to x (or from x to a if $x < a$).

We also need to assume that for $\int_a^x 1/t\,dt$ to make sense, and so perhaps you can say that the above solution must be right because the question of finding the derivative does not make sense if 0 is between a and x.

3.1.4 Theorem (Other part of fundamental theorem)**.** *Assume that $y = f(x)$ is continuous for $a \le x \le b$ and suppose $g(x)$ is an antiderivative for $f(x)$ (that is $g'(x) = f(x)$ for $a \le x \le b$). Then*

$$\int_a^b f(x)\,dx = g(b) - g(a) = [g(x)]_{x=a}^b$$

This is familiar. For instance:

3.1.5 Examples. (i) Find $\displaystyle\int_1^2 e^{2x}\,dx$

Solution: For $f(x) = e^{2x}$, we need a $g(x)$ with $g'(x) = f(x)$ (an antiderivative). We can 'guess' an answer if we notice that

$$\frac{d}{dx}e^{2x} = e^{2x}\frac{d}{dx}(2x) = e^{2x}(2) = 2e^{2x}$$

and so

$$\frac{d}{dx}\frac{1}{2}e^{2x} = e^{2x}$$

That is $g(x) = (1/2)e^{2x}$ works. So

$$\int_1^2 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_1^2 = \frac{1}{2}e^4 - \frac{1}{2}e^2$$

(ii) Find $\displaystyle\int_1^4 \frac{1}{x}\,dx$

Solution: We learned recently that $\ln x$ has derivative $1/x$ for $x > 0$. So for $f(x) = 1/x$ (with $x > 0$) we can take $g(x) = \ln x$. We get

$$\int_1^4 \frac{1}{x}\,dx = [\ln x]_1^4 = \ln 4 - \ln 1$$

We can simplify that because $\ln 1 = 0$. So

$$\int_1^4 \frac{1}{x}\,dx = \ln 4$$

Aside: That would have worked with 4 replaced by any positive value and in fact

$$\int_1^b \frac{1}{x}\,dx = \ln b \text{ if } b > 0$$

Switching around some letters, that means

$$\int_1^x \frac{1}{t}\,dt = \ln x \text{ if } x > 0$$

An alternative approach to where the functions e^x and its inverse $\ln x$ come from is to start by defining $\ln x = \int_1^x \frac{1}{t}\,dt$, use the form Theorem 3.1.1 of the Fundamental Theorem to show $(d/dx)\ln x = 1/x$ for $x > 0$, and then derive the other properties of $\ln$. Finally get to the exponential as the inverse. This is done in Anton §6.6.

Notice that to use Theorem 3.1.4 to evaluate an integral, you need a different way to find an antiderivative $g(x)$ for the integrand $f(x)$ than Theorem 3.1.1 or Corollary 3.1.2. If you use Theorem 3.1.1 to find $g(x) = A(x)$, and then use Theorem 3.1.4, you end up saying that the integral is equal to itself.

3.2 Antiderivatives we know

(Anton §7.1.)

At the end result of our studies will be that we will be able to find a certain number of integrals using the techniques we will describe. But, unlike differentiation where we can differentiate almost anything we can write down using the basic rules (including the chain rule, product rule and quotient rule), with integration it is easy to come across simple-looking things we will not be able to do. One example is $\int \sin(x^2)\,dx$.

For each differentiation formula, we have a corresponding integration formula. Here are ones we know (mostly).

Derivative formula	**Integration formula**
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}x^a = ax^{a-1}\ (x > 0)$	$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C$ if $a \neq -1$, $x > 0$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$

Derivative formula	**Integration formula**
$\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}\ (-1 < x < 1)$	$\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$	$\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\dfrac{d}{dx}\sinh x = \cosh x$	$\displaystyle\int \cosh x\,dx = \sinh x + C$
$\dfrac{d}{dx}\cosh x = \sinh x$	$\displaystyle\int \sinh x\,dx = \cosh x + C$
$\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$	$\displaystyle\int \operatorname{sech}^2 x\,dx = \tanh x + C$
$\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{1+x^2}}$	$\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$ $= \ln(x + \sqrt{x^2+1}) + C$
$\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}\ (x > 1)$	$\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$ $= \ln(x + \sqrt{x^2-1}) + C$
$\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$	$\displaystyle\int \frac{1}{1-x^2}\,dx = \tan^{-1} x + C$ $= \dfrac{1}{2}\ln\dfrac{1+x}{1-x} + C\ (-1 < x < 1)$ $\displaystyle\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left\lvert\frac{1+x}{1-x}\right\rvert + C$

The last of these formulae is not quite one we knew already. It works for $x < -1$, for $-1 < x < 1$ and for $x > 1$. Also the one that says $\int (1/x)\,dx = \ln|x| + C$ needs some explanation. It is true for $x > 0$ (we know $(d/dx)\ln x = 1/x$ if $x > 0$ and $|x| = x$ if $x > 0$. For $x < 0$, we have $|x| = -x$ and then

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = x \qquad (x < 0).$$

So the formula is correct for either case $x > 0$ or $x < 0$ but in fact it is not permissible to cross from $x > 0$ to $x < 0$ in the same discussion (because $1/x$ does not make sense at $x = 0$ and definite integrals of $1/x$ also won't make sense if $x = 0$ is included in the integration).

Apart from these basic integrals, there are integration formulae that follow from the chain rule for differentiation and the product rule for differentiation. Essentially, when we rearrange the integral of the two formulae, we get the methods of substitution and integration by parts.

3.3 Substitution (reminder)

(Anton §4.3.)

Last semester you studied the technique of substitution, which is in essence based on the chain rule. We recall some simple examples:

3.3.1 Examples. (i) Consider

$$\int e^{2x}\,dx$$

(which arose earlier and we solved more or less by 'guessing'). A simple substitution that works here is $u = 2x$, leading to $du = 2\,dx$, or $dx = \frac{1}{2}\,du$ and

$$\int e^{2x}\,dx = \int e^u \frac{1}{2}\,du = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x} + C$$

(ii) Consider the slightly more complicated problem of finding

$$\int x^2\cos(x^3)\,dx$$

Solution: If we notice that $u = x^3$ has $\frac{du}{dx} = 3x^2$ and is a factor in the integrand apart from the constant factor 3, then we can write

$$\begin{aligned}
\int x^2\cos(x^3)\,dx & & \text{Let } u = x^3\\
& & \frac{du}{dx} = 3x^2\\
& & du = 3x^2\,dx\\
& & \frac{du}{3x^2} = dx\\
\int x^2\cos(x^3)\,dx &= \int 3x^2\cos u\,\frac{du}{3x^2}\\
&= \int \frac{1}{3}\cos u\,du\\
&= \frac{1}{3}\sin u + C\\
&= \frac{1}{3}\sin(x^3) + C
\end{aligned}$$

For a substitution to work, it is usually necessary for du/dx to be a factor in the integrand (essentially, maybe in a slightly hidden way). It is vital that we express all x's and dx's in terms of u and du.

3.3.2 Example. For a definite integral like $\int_2^5 x^2 \cos(x^3)\,dx$, where substitution is useful, we can work out the indefinite integral as above and then put in the limits, **or** we can change to limits for u like this

$$\int_2^5 x^2 \cos(x^3)\,dx = \int_{u=2^3}^{u=5^3} \frac{1}{3}\cos u\,du = \int_{u=8}^{u=125} \frac{1}{3}\cos u\,du = \left[\frac{1}{3}\sin u\right]_{u=8}^{u=125} = \frac{1}{3}\sin 125 - \frac{1}{3}\sin 8$$

To reiterate, if we make a substitution $u = u(x)$ in a definite integral $\int_a^b f(x)\,dx$ we can change it to a new definite integral where the limits are the limits for u that correspond to $x = a$ and $x = b$:

$$\int_a^b f(u(x))\,dx = \int_{x=a}^{x=b} f(u(x))\,dx = \int_{u=u(a)}^{u=u(b)} f(u)\frac{dx}{du}\,du$$

(and this has the advantage that is says one number is equal to another — no need to go back and express u in terms of x when you get the antiderivative, something that can be messy in some examples).

3.4 Integration by parts

(Anton §7.2.)

If we integrate both sides of the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

we get

$$\int \frac{d}{dx}(uv)\,dx = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

or

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

This allows us a way of transforming integrals that take the form of a product of one function times the derivative of another

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

into a different integral (where the differentiation has flipped from one factor to the other). The advantage of this comes if we know how to manage the new integral (or at least if it is simpler than the original). The integration by parts formula is usually written with the dx's cancelled

$$\boxed{\int u\,dv = uv - \int v\,du}$$

3.4.1 Examples. (a) $\int x \ln x \, dx$

Solution: The two most obvious ways to use integration by parts are

- $u = x$, $dv = \ln x \, dx$ (Problem with this is we can't find v very easily)
- $u = \ln x$, $dv = x \, dx$

It turns out that the second is good.

$$\begin{aligned}
\int x \ln x \, dx \qquad & \text{Let} \quad \begin{array}{rclrcl} u & = & \ln x & dv & = & x\,dx \\ du & = & \frac{1}{x}\,dx & v & = & \frac{x^2}{2} \end{array} \\
\int x \ln x \, dx &= \int u \, dv \\
&= uv - \int v \, du \\
&= (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx \\
&= (x^2/2)\ln x - \int \frac{x}{2}\,dx \\
&= (x^2/2)\ln x - \frac{x^2}{4} + C
\end{aligned}$$

(b) $\int_1^e \ln x \, dx$

Solution: This is one of a very few cases which can be done by taking $dv = dx$ and $u =$ the integrand. Every integral takes the form $\int u \, dv$ in that way, but it is rarely a good way to start integration by parts.

Here we can make use of the definite integral form of the integration by parts formula

$$\boxed{\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du}$$

which arises in the same way as the indefinite integral formula (take definite integrals of the product rule for differentiation).

$$\int_1^e \ln x\,dx \qquad \text{Let} \quad \begin{array}{rcl} u &=& \ln x \\ du &=& \frac{1}{x}\,dx \end{array} \quad \begin{array}{rcl} dv &=& dx \\ v &=& x \end{array}$$

$$\begin{aligned} \int_1^e \ln x\,dx &= \int_1^e u\,dv \\ &= [uv]_1^e - \int_1^e v\,du \\ &= [(\ln x)x]_1^e - \int_1^e x\frac{1}{x}\,dx \\ &= e\ln e - \ln 1 - \int_1^e 1\,dx \\ &= e - [x]_1^e \\ &= e - (e-1) = 1 \end{aligned}$$

(c) $\int x^2 \cos x\,dx$

Solution:

$$\int x^2 \cos x\,dx \qquad \text{Let} \quad \begin{array}{rcl} u &=& x^2 \\ du &=& 2x\,dx \end{array} \quad \begin{array}{rcl} dv &=& \cos x\,dx \\ v &=& \sin x \end{array}$$

$$\begin{aligned} \int x^2 \cos x\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x^2 \sin x - \int \sin x(2x)\,dx \\ &= x^2 \sin x - \int 2x \sin x\,dx \end{aligned}$$

The point here is that we have succeeded in simplifying the problem. We started with x^2 times a trigonometric function ($\cos x$) and we have now got to x times a trigonometric function ($\sin x$ this time, but that is not so different in difficulty to $\cos x$). If we continue in the *same* (or similar) way and apply integration by parts again, we can make the problem even simpler. We use U and V this time in case we might get confused with the earlier u and v.[1]

$$\int 2x\sin x\,dx \qquad \text{Let} \quad \begin{array}{rcl} U &=& 2x \\ dU &=& 2\,dx \end{array} \quad \begin{array}{rcl} dV &=& \sin x\,dx \\ V &=& -\cos x \end{array}$$

$$\begin{aligned} \int 2x\sin x\,dx &= \int U\,dV \\ &= UV - \int V\,dU \\ &= 2x(-\cos x) - \int (-\cos x)2\,dx \\ &= -2x\cos x + \int 2\cos x\,dx \\ &= -2x\cos x + 2\sin x + C \end{aligned}$$

Combining with the first stage of the calculation

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x - C$$

and, in fact $-C$ is plus another constant. Since C can be any constant, the answer

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$$

is also good.

3.4.2 Remark. We will not in fact learn any other techniques than these which are purely integration methods. We will spend some time explaining how to make use of these techniques in specific circumstances (as it is often not at all obvious how to do so). There is one other method we will come to called *partial fractions*, a method for integrating fractions such as

$$\int \frac{x+2}{(x-1)(x^2+2x+2)}\,dx$$

However, the thing we have to learn about is algebra — a way to rewrite fractions like this as sums of simpler ones — and there is no new idea that is directly integration. The algebra allows us to tackle problems of this sort.

3.5 Trigonometric Integrals

(Anton §7.3.)

(i) **Powers of $\cos x$ times powers of $\sin x$ with one power odd**

Method: For

$$\int \sin^n x\cos^m x\,dx$$

[1] One thing to avoid is $U = v$ and $V = u$ because this will just unravel what we did to begin with.

- if $n =$ the power of $\sin x$ is odd, substitute $u = \cos x$
- if $m =$ the power of $\cos x$ is odd, substitute $u = \sin x$

3.5.1 Example. $\int \sin^3 x \cos^4 x \, dx$

Solution: Let $u = \cos x$, $du = -\sin x \, dx$, $dx = \dfrac{du}{-\sin x}$

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \sin^3 x u^4 \frac{du}{-\sin x} \\
&= \int -\sin^2 x u^4 \, du \\
&= \int -(1-\cos^2 x)u^4 \, du \\
&= \int -(1-u^2)u^4 \, du \\
&= \int u^6 - u^4 \, du \\
&= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\
&= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

(ii) **Powers of $\cos x$ times powers of $\sin x$ with both powers even**

Method: use the trigonometric identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

3.5.2 Example. $\int \sin^4 x \cos^2 x \, dx$

Solution:

$$\begin{aligned}
\int \sin^4 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \, dx \\
&= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \left(\frac{1}{2}(1+\cos 2x)\right) dx \\
&= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x)(1+\cos 2x) \, dx \\
&= \frac{1}{8}\int 1 - \cos 2x - \cos^2 2x + \cos^3 2x \, dx
\end{aligned}$$

Now $\int 1\,dx$ is no bother. $\int \cos 2x\,dx$ is not much harder than $\int \cos x\,dx = \sin x + C$. If we look at

$$\frac{d}{dx}\sin 2x = (\cos 2x)2$$

we can see that $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. (This can also be done by a substitution $u = 2x$ but that is hardly needed.) Next

$$\int \cos^2 2x\,dx = \int \frac{1}{2}(1+\cos 4x)\,dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C$$

(using the same ideas as for $\int \cos 2x\,dx$).

For $\int \cos^3 2x\,dx$ we are in a situation where we have an odd power of $\cos$ times a zeroth power of $\sin$. So we can use the earlier method (the fact that the angle is $2x$ doe snot make a big difference) of substituting $u = \sin 2x$. Then $du = 2\cos 2x\,dx$, $dx = \dfrac{du}{2\cos 2x}$,

$$\begin{aligned}
\int \cos^3 2x\,dx &= \int \cos^3 2x \frac{du}{2\cos 2x} \\
&= \frac{1}{2}\int \cos^2 2x\,du \\
&= \frac{1}{2}\int 1 - \sin^2 2x\,dx \\
&= \frac{1}{2}\int 1 - u^2\,du \\
&= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C \\
&= \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C
\end{aligned}$$

Putting all the bits together

$$\begin{aligned}
\int \sin^4 x\cos^2 x\,dx &= \frac{1}{8}\int 1 - \cos 2x - \cos^2 2x + \cos^3 2x\,dx \\
&= \frac{1}{8}\left(x - \frac{1}{2}\sin 2x - \frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right) + C \\
&= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C
\end{aligned}$$

(iii) **Powers of $\sin x$ and $\cos x$**

Method: Use the previous two methods treating

$$\int \sin^n x\,dx = \int \sin^n x(\cos x)^0\,dx$$

and similarly for $\int \cos^m x\,dx$ (that is treat the second power as the zeroth power).

3.5.3 Examples. • $\int \cos^3 x\,dx$

Solution: $\int \cos^3 x\,dx = \int (\sin x)^0 \cos^3 x\,dx$. Power of $\cos$ odd. Substitute $u = \sin x$, $du = \cos x\,dx$, $dx = \dfrac{du}{\cos x}$

$$\begin{aligned}
\int \cos^3 x\,dx &= \int \cos^3 x\,\frac{du}{\cos x} \\
&= \int \cos^2 x\,du \\
&= \int 1 - u^2\,du \\
&= u - \frac{1}{3}u^3 + C \\
&= \sin x = \frac{1}{3}\sin^3 x + C
\end{aligned}$$

• $\int \sin^4 x\,dx$

Solution: Use even powers method.

$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx \\
&= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2\,dx \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,dx \\
&\qquad \text{Note: still have one even power} \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\,dx \\
&= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

3.6 Inverse trigonometric substitutions

(Anton §7.4.)

We now consider a class of substitutions which seem quite counter intuitive.

Recall these

$$
\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}}\\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2}\\
\frac{d}{dx}\cosh^{-1}x &= \frac{1}{\sqrt{x^2-1}}\\
\frac{d}{dx}\sinh^{-1}x &= \frac{1}{\sqrt{x^2+1}}
\end{aligned}
$$

The corresponding substitution methods are:

- integrals involving $\sqrt{1-x^2}$, substitute $\theta = \sin^{-1} x$ (or it is often more convenient to write it $x = \sin\theta$).

  More generally, integrals involving $\sqrt{a^2-u^2}$ (with a constant) — try substituting $u = a\sin\theta$ (or $\theta = \sin^{-1}(u/a)$).

- integrals involving $\dfrac{1}{a^2+u^2}$ — try substituting $u = a\tan\theta$
- integrals involving $\sqrt{u^2-a^2}$ — try substituting $u = a\cosh t$
- integrals involving $\sqrt{u^2+a^2}$ — try substituting $u = a\sinh t$

3.6.1 Examples. (i) $\displaystyle\int \sqrt{9-4(x+2)^2}\,dx$

Solution: In this case, we have $\sqrt{a^2-u^2}$ with $a^2 = 9$, $a = 3$, $u = 2(x+2)$ and so our method says to try $u = a\sin\theta$ or

$$
\begin{aligned}
2(x+2) &= 3\sin\theta\\
2\,dx &= 3\cos\theta\,d\theta\\
dx &= \frac{3\cos\theta}{2}\,d\theta\\
\int \sqrt{9-4(x+2)^2}\,dx &= \int \sqrt{9-9\sin^2\theta}\frac{3\cos\theta}{2}\,d\theta\\
&= \int \sqrt{9\cos^2\theta}\frac{3\cos\theta}{2}\,d\theta\\
&= \int 3\cos\theta\frac{3\cos\theta}{2}\,d\theta\\
&= \frac{9}{2}\int \cos^2\theta\,d\theta\\
&= \frac{9}{4}\int 1+\cos 2\theta\,d\theta\\
&= \frac{9}{4}(\theta + \frac{1}{2}\sin 2\theta) + C
\end{aligned}
$$

To get the answer in terms of x, we need θ in terms of x

$$\begin{aligned} 2(x+2) &= 3\sin\theta \\ \frac{2}{3}(x+2) &= \sin\theta \\ \theta &= \sin^{-1}\left(\frac{2}{3}(x+2)\right) \end{aligned}$$

and we could get a correct answer by replacing θ by this everywhere in the answer above.

There is a way to simplify the answer but we won't go into that.

(ii) $\int \sqrt{-4x^2 - 16x - 7}\, dx$

Solution: For quadratics inside a square root like this, what we should do first is complete the square. That is, rearrange the x^2 and x terms so that (with a suitable constant) we get a multiple of a perfect square

$$\begin{aligned} -4x^2 - 16x - 7 &= -4(x^2 + 4x) - 7 \\ &= -4(x^2 + 4x + 4 - 4) - 7 \\ &= -4(x^2 + 4x + 4) + 9 \\ &= 9 - 4(x+2)^2 \end{aligned}$$

This means that not only is this problem similar to the previous one, it is in fact the same problem again (now that we completed the square).

3.6.2 Remark. There are in fact many more tricks we could go into.

3.7 Partial Fractions

(Anton §7.5.)

Partial fractions are a technique from algebra, but our reason for dealing with them is that they can in principle help find every integral of the form

$$\int \frac{p(x)}{q(x)}\, dx$$

where $p(x)$ and $q(x)$ are polynomials.

Except in a few special cases, we don't yet know how to find such integrals. One special case, where we don't need partial fractions, is where $q'(x) = p(x)$ or $q'(x) = \alpha p(x)$ for some constant α, because in these cases we can make a substitution $u = q(x)$, $du = q'(x)\, dx$ and it will work out nicely. In fact substitution would also work if $q(x) = r(x)^n$ for some $n \geq 1$ and $r'(x) = \alpha p(x)$ for a constant α — we can substitute $u = r(x)$, $du = r'(x)\, dx = \alpha p(x)\, dx$,

$$\int \frac{p(x)}{q(x)}\, dx = \int \frac{p(x)}{r(x)^n}\, dx = \int \frac{p(x)}{u^n} \frac{du}{\alpha p(x)} = \int (1/\alpha)\frac{1}{u^n}\, du$$

The idea of partial fractions is to rewrite $\frac{p(x)}{q(x)}$ as a sum of fractions with simple denominators and numerators that are somehow small compared to the denominator. We need to explain exactly how it goes.

We need to talk about factoring polynomials as much as possible.

To start with, a polynomial is an expression you get by taking a finite number of powers of x with constant coefficients in front and adding them up. For example

$$p(x) = 4x^2 - x + 17$$

or

$$p(x) = 27x^{11} + 15x^{10} - x^9 + x^8 + 11x^2 + 5$$

are polynomials. The highest power of x that has a nonzero coefficient in front is called the *degree* of the polynomial. The examples above have degree 2 and degree 11.

What is handy to know is that when we multiply polynomials, the degrees add. So $(x + 1)(x + 5)(x^2 + x + 11)$ will have degree $1 + 1 + 2 = 4$ when it is multiplied out. Constant polynomials have degree 0, except the zero polynomial — we are best not giving any degree to the zero polynomial.

Now, what kind of polynomial can be factored? For this purpose we don't consider constant factors as genuine factors. So

$$2x^2 + 4 = 2(x^2 + 2) = \frac{1}{3}(6x^2 + 12)$$

will not be counted as a factorization.

Anything of degree 1 certainly cannot be factored then. Some things of degree 2 can be factored, such as

$$x^2 + 5x + 4 = (x + 1)(x + 4)$$

but other quadratics cannot be factored if we don't allow complex numbers to be used. We *cannot factor*

$$x^2 + 2x + 2 = (x - \alpha)(x - \beta)$$

because if we could then the roots of $x^2 + 2x + 2$ would be α and β. The roots of $x^2 + 2x + 2$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

and these are complex numbers. So α and β would have to be these complex numbers.

A remarkable fact is that every polynomial of degree 3 or more **can** be factored, at least in theory. It does not mean it is easy to find the factors, unfortunately. What you can sometimes rely on to factor polynomials is the **Remainder Theorem.** Recall that it says that if $p(x)$ is a polynomial and you know a root $x = a$ (that is a value a so that $p(a) = 0$), then $x - a$ must divide $p(x)$.

Using the theory, just as in principle whole numbers can be factored as a product of prime numbers, so polynomials $p(x)$ with real coefficients can be factored as a product of linear factors like $x - a$ and quadratic factors $x^2 + bx + c$ with complex roots. If the coefficient of the highest power of x in $p(x)$ is not 1, then we also need to include that coefficient. So a complete factorization of

$$2x^2 + 8x + 2 = (2x + 4)(x + 2) = 2(x + 2)(x + 2) = 2(x + 2)^2.$$

For $3x^3 + 3x^2 + 6x + 6 = 3(x^3 + x^2 + 2x + 2)$, you can check that $x = -1$ is a root and so $x - (-1) = x + 1$ must divide it. We get

$$3x^3 + 3x^2 + 6x + 6 = 3(x + 1)(x^2 + 2)$$

from long division.

Now we can outline **how partial fractions works** for a fraction $\frac{p(x)}{q(x)}$ of two polynomials:

Step 0: (preparatory step). If the degree of the numerator $p(x)$ is not strictly smaller than the degree of the denominator $q(x)$, use long division to divide $q(x)$ into $p(x)$ and obtain a quotient $Q(x)$ and remainder $R(x)$. Then

$$\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$$

and degree$(R(x)) <$ degree$(q(x))$.

We concentrate then on the 'proper fraction' part $R(x)/q(x)$.

Step 1: Now factor $q(x)$ *completely* into a product of linear factors $x - a$ and quadratic factors $x^2 + bx + c$ with complex roots.

Gather up any repeated terms, so that if (say) $q(x) = (x - 1)(x + 2)(x^2 + 3)(x + 2)$ we would write it as $q(x) = (x - 1)(x + 2)^2(x^2 + 3)$.

Step 2: Then the proper fraction $\dfrac{R(x)}{q(x)}$ can be written as a sum of fractions of the following types:

(i) $\dfrac{A}{(x - a)^m}$

(ii) $\dfrac{Bx + C}{(x^2 + bx + c)^k}$

where we include all possible powers $(x - a)^m$ and $(x^2 + bx + c)^k$ that divide $q(x)$. The A, B, C stand for constants.

As examples, consider the following. We just write down what the partial fractions look like. In each case, we start with a proper fraction where the denominator is completely factored already. So some of the hard work is already done.

(i) $\dfrac{x^2 + x + 5}{(x - 1)(x - 2)(x - 3)} = \dfrac{A_1}{x - 1} + \dfrac{A_2}{x - 2} + \dfrac{A_3}{x - 3}$

(ii) $\dfrac{x^3+2x+7}{(x+1)^2(x-4)} = \dfrac{A_1}{x+1} + \dfrac{A_2}{(x+1)^2} + \dfrac{A_3}{x-4}$

(iii) $\dfrac{x^2+x+11}{(x+1)(x^2+2x+2)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2x+2}$

(iv) $\dfrac{x^4+x+11}{(x+1)(x^2+2x+2)^2} = \dfrac{A}{x+1} + \dfrac{B_1x+C_1}{x^2+2x+2} + \dfrac{B_2x+C_2}{(x^2+2x+2)^2}$

To make use of these, we have to be able to find the numbers $A, B, C, \ldots$ that make the equation true. Take the first example

$$\frac{x^2+x+5}{(x-1)(x-2)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

To find the appropriate A_1, A_2, A_3, we multiply across by the original denominator $(x-1)(x-2)(x-3)$. This has the effect of clearing all the fractions.

$$\begin{aligned} x^2+x+5 &= \frac{A_1}{x-1}(x-1)(x-2)(x-3) + \frac{A_2}{x-2}(x-1)(x-2)(x-3) \\ &\quad + \frac{A_3}{x-3}(x-1)(x-2)(x-3) \\ &= A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2) \end{aligned}$$

There are two avenues to pursue from here. In this case, method 1 seems easier to me, but in general method 2 can be as good.

Method 1. Plug in the values of x that make the original denominator $(x-1)(x-2)(x-3) = 0$.

$$\begin{aligned} x = 1 &: \\ 1+1+5 &= A_1(1-2)(1-3) + A_2(0) + A_3(0) \\ 7 &= 2A_1 \\ & \boxed{A_1 = 7/2} \\ x = 2 &: \\ 11 &= A_1(0) + A_2(1)(-1) + A_3(0) \\ &= -A_2 \\ & \boxed{A_2 = -11} \\ x = 3 &: \\ 17 &= 0 + 0 + A_3(2)(1) \\ & \boxed{A_3 = 17/2} \end{aligned}$$

So we get

$$\frac{x^2+x+5}{(x-1)(x-2)(x-3)} = \frac{7/2}{x-1} + \frac{-11}{x-2} + \frac{17/2}{x-3}.$$

Our interest in this is for integration. We can now easily integrate

$$\begin{aligned}\int \frac{x^2+x+5}{(x-1)(x-2)(x-3)}\,dx &= \int \frac{7/2}{x-1}+\frac{-11}{x-2}+\frac{17/2}{x-3}\,dx \\ &= \frac{7}{2}\ln|x-1| - 11\ln|x-2| + \frac{17}{2}\ln|x-3| + C\end{aligned}$$

Method 2. Multiply out the right hand side.

$$\begin{aligned}x^2+x+5 &= A_1(x-2)(x-3)+A_2(x-1)(x-3)+A_3(x-1)(x-2) \\ &= A_1(x^2-5x+6)+A_2(x^2-4x+3)+A_3(x^2-2x+2) \\ &= (A_1+A_2+A_3)x^2+(-5A_1-4A_2-2A_3)x+(6A_1+3A_2+2A_2)\end{aligned}$$

and compare the coefficients on both sides to get a system of linear equations

$$\begin{array}{rcrcrcl} A_1 & + & A_2 & + & A_3 & = & 1 \\ -5A_1 & - & 4A_2 & - & 2A_3 & = & 1 \\ 6A_1 & + & 3A_2 & + & 2A_2 & = & 5 \end{array}$$

These can be solved (by Gaussian elimination, for example) to find A_1, A_2, A_3.

Method 1 is certainly magic in this case, but there are examples where Method 1 does not get all the unknown so easily.

Another example. Here is one of our previous examples with the numbers worked out.

$$\frac{x^2+x+11}{(x+1)(x^2+2x+2)} = \frac{11}{x+1} + \frac{-10x-11}{x^2+2x+2}$$

To find the integral of this,

$$\begin{aligned}\int \frac{x^2+x+11}{(x+1)(x^2+2x+2)}\,dx &= \int \frac{11}{x+1}\,dx - \int \frac{10x+11}{x^2+2x+2}\,dx \\ &= 11\ln|x+1| - \int \frac{10x+11}{x^2+2x+2}\,dx\end{aligned}$$

To work out the remaining integral, we use the method of completing the square $x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$ and there is a trick. The trick is inspired by the fact that the substitution $u = x^2+2x+2$, $du = (2x+2)\,dx = 2(x+1)\,dx$ would work if the numerator was a multiple of $x+1$. What we can do is write

$$\int \frac{10x+11}{x^2+2x+2}\,dx = \int \frac{10x+10}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

and make the $u = x^2+2x+2$ substitution in the first half, while the second is an inverse $\tan$ example. (By substituting $w = x+1$, $dw = dx$, the second part becomes

$$\int \frac{1}{(x+1)^2+1}\,dx = \int \frac{1}{w^2+1}\,dw = \tan^{-1} w = \tan^{-1}(x+1)$$

or we might be able to guess that.) We get

$$\begin{aligned}\int \frac{10x+11}{x^2+2x+2}\,dx &= \int \frac{10x+10}{u}\frac{du}{2(x+1)} + \int \frac{1}{(x+1)^2+1}\,dx \\ &= \int \frac{5}{u}\,du + \tan^{-1}(x+1) \\ &= 5\ln|u| + \tan^{-1}(x+1) + C \\ &= 5\ln(x^2+2x+2) + \tan^{-1}(x+1) + C\end{aligned}$$

Finally, our integral works out as

$$\int \frac{x^2+x+11}{(x+1)(x^2+2x+2)}\,dx = 11\ln|x+1| - 5\ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

3.8 Improper integrals

(Anton §7.8.)

3.8.1 Remark. Sometimes, integrals that appear to be infinite in extent can be given a finite value in a way that seems sensible.

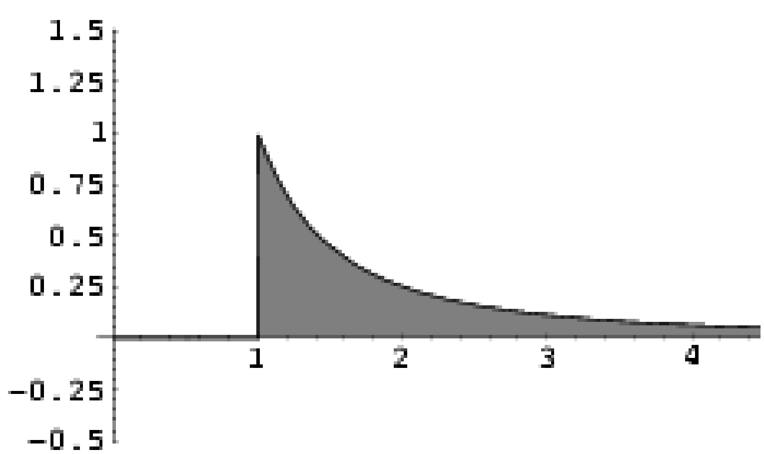
For example, consider

$$\int_1^\infty \frac{1}{x^2}\,dx$$

If we draw a picture of what this might mean graphically, in the same way as we did for integrals $\int_a^b f(x)\,dx$ where a and b are finite, we should be looking at the area of the region under the graph

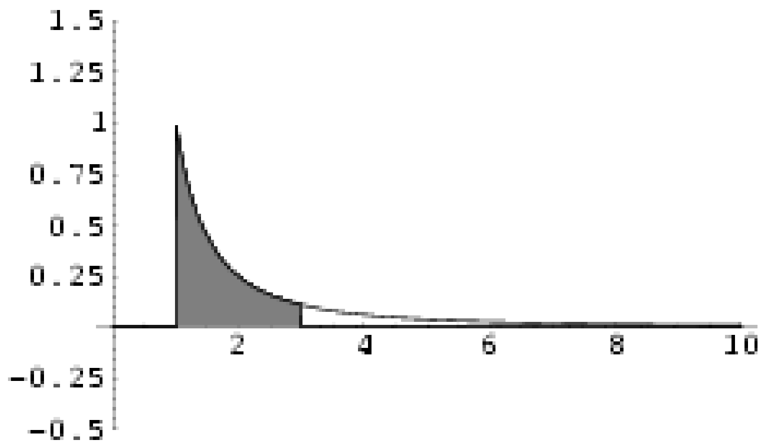
$$y = \frac{1}{x^2} \qquad x \geq 1$$

— a region that stretches infinitely far into the distance. So it seems infinite and nothing more to be said.



But, before we conclude that it is infinite, suppose we imagine colouring in the area under that graph with paint, and we do it so that we apply the paint evenly so that we use a fixed amount per square inch. The amount of paint we would need should be infinite if the area is infinite.

We would never be done painting an infinite area, an so we could paint a wide but finite section and see how much paint we need.



In this picture, we show colouring for $1 \leq x \leq 3$, but we could replace 3 by any finite $b > 1$. The area covered up as far as b works out as

$$\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = -\frac{1}{b} - (-1) = 1 - \frac{1}{b}$$

and we see that, rather than a huge answer, we always get an answer < 1. In fact as $b \to \infty$, the answer approaches 1. It does not tend to ∞. So if we have enough paint to paint 1 square unit of area, we will never completely run out although there will be very little paint left when b is large.Maybe there is a case for deciding that $\int_1^\infty 1/x^2\,dx$ should have the value 1.

We make this our *definition*.

3.8.2 Definition. We *define*

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

(which is 1).

More generally, if $y = f(x)$ is defined and continuous for $x \geq a$ we *define the improper integral*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$$

if this limit exists and is *finite*. On the other hand if the limit does not exist at all, or is infinite, we say that the improper integral $\int_a^\infty f(x)\,dx$ fails to converge.

3.8.3 Example. Find $\int_2^\infty 1/x\,dx$.

In this kind of example (an improper integral) we start by using the right definition. This shows that we realize that there is an issue about the integral making sense, and that we know how the issue is dealt with. We get

$$\begin{aligned}\int_2^\infty 1/x\,dx &= \lim_{b\to\infty}\int_2^b 1/x\,dx\\ &= \lim_{b\to\infty}[\ln x]_2^b\\ &= \lim_{b\to\infty}\ln b - \ln 2.\end{aligned}$$

If we look at the graph of $y = \ln x$ we see that this limit is ∞. Because it is not finite, we say that the improper integral

$$\int_2^\infty \frac{1}{x}\,dx$$

does not converge.

Other types of improper integral

Integrals can be improper because of vertical asymptotes, or because the area under the graph stretches infinitely far away in different ways.

3.8.4 Definition. (i) If $y = f(x)$ is defined and continuous for $x \le b$, then we define the improper integral

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral $\int_{-\infty}^b f(x)\,dx$ does not converge.

(ii) If $y = f(x)$ is continuous on a finite interval $a < x \le b$, excluding the left end point a (where it might have an asymptote or other bad behaviour), then we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral does not converge.

Example. Consider $\int_0^1 1/x^2\,dx$.

Using the definition

$$\begin{aligned}\int_0^1 1/x^2\,dx &= \lim_{c\to0^+}\int_c^1 1/x^2\,dx\\ &= \lim_{c\to0^+}\left[-\frac{1}{x}\right]_c^1\\ &= \lim_{c\to0^+} -1 + \frac{1}{c}\end{aligned}$$

As this limit is infinite, the improper integral $\int_0^1 1/x^2\,dx$ does not converge.

(iii) If $y = f(x)$ is continuous on a finite interval $a \leq x < b$, excluding the right end point b (where it might have an asymptote or other bad behaviour), hen we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

if this limit exists and is finite. If the limit does not exist, or is infinite, we say that the improper integral does not converge.

(iv) If an integral is improper for more than one reason, or at a place not one of the endpoints, we divide it as a sum of improper integrals of the types we have considered above. If each one of the bits has a finite value, then the value of the whole is the sum. But if any one of the bits fails to converge then the whole is said not to converge.

3.8.5 Examples. (a) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

Here the problem with the integral is at 0, where the integrand has an asymptote. In fact we already worked out that $\int_0^1 1/x^2\,dx$ does not converge and so we know that $\int_{-1}^1 \frac{1}{x^2}\,dx$ also does not converge.

This is an example where an unthinking use of integration would produce a wrong answer -2. (You might be suspicious if the integral of a positive thing in the left to right direction turned out to be negative.)

(b) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

(It does not matter about using 0 as a stopping point. Any finite point would do, for example $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^4 \frac{1}{1+x^2}\,dx + \int_4^{\infty} \frac{1}{1+x^2}\,dx$)

We have an antiderivative $\tan^{-1} x$ for $\frac{1}{1+x^2}$ and so we can evaluate the first integral easily

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \int_a^0 \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} [\tan^{-1} x]_{-a}^0 = \lim_{a \to -\infty} (0 - \tan^{-1}(-a)) = \frac{\pi}{2}$$

The other integral $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ also works out at $\pi/2$.

(c) $$\int_{-\infty}^{\infty} \frac{1}{x(1+x^2)}\,dx = \int_{-\infty}^1 \frac{1}{x(1+x^2)}\,dx + \int_{-1}^0 \frac{1}{x(1+x^2)}\,dx + \int_0^1 \frac{1}{x(1+x^2)}\,dx + \int_1^{\infty} \frac{1}{x(1+x^2)}\,dx$$

The original is improper because of the $-\infty$ limit, the asymptotes at $x = 0$ and the ∞ limit. Each of the 4 bits above has just one problem at one end (and needs to be worked out as a limit).

3.9 Using computers to find integrals

We have now concluded our relatively superficial study of methods of finding indefinite integrals. There is an art to to being able to use the methods well, but really there are not so many more such methods.

However, as discussed in Anton, there are some other approaches. In §7.6 Anton has a section called "Using computer algebra systems and tables of integrals". Tables of integrals have pretty much become obsolete now, but they are like the lists we have in §3.2 above or page 26 of the "formulae & tables" produced by the State Examinations Commission — only much longer lists. You use the tables by trying to spot something in the list that seems to help with your particular problem.

There are various computer systems that can do this (or something more clever) for you. The most commercially successful is Mathematica (produced by Wolfram research) and it can do most calculations you can do by hand (and more). There is a website `wolframalpha.com` which provides a kind of Scientific version of Google searches and uses Mathematica behind the scenes. It is perhaps easier to use than Mathematica itself, as Mathematica wants you to follow its rules and complains if you don't (a bit like WileyPlus in that respect). However `wolframalpha.com` tries to give an answer no matter what you ask it and so it can be hard to control it to find the specific answer you want.

Another computer algebra system like Mathematica is called (Waterloo) Maple and it is also commercial. There is a free thing called SageMath and you can download it (it is very big) or use it online (see `http://www.sagemath.org/`). It probably takes a bit of learning before it would be of use at all.

What is relatively new about computer algebra systems are that (a) they can compute with symbols, not just with numerical values and (b) modern computers such as your laptop are powerful enough for these systems. So they are in a sense accessible (though several of them are expensive — however see also
`http://www.tcd.ie/itservices/software/kb/student_software.php`).

Computing with numbers is what computers were doing for decades after they were invented. They can still do that, your calculator does it on a small scale, and this sort of use is important for engineers who will now frequently simulate a machine numerically before attempting to build it.

One aspect of such simulations often comes down to working out definite integrals. How can a machine calculate a definite integral?

In theory, this is easy. A definite integral $\int_a^b f(x)\,dx$ is defined as a limit of some finite sums called "Riemann sums". Assuming a and b are known exact numbers and $f(x)$ is given by a formula that the computer can evaluate when fed a value for x, the computer can compute a Riemann sum (with lots of strips, or a fine subdivision of $[a, b]$). That will give an approximate value for $\int_a^b f(x)\,dx$. If the computer does the same calculation more times with finer and finer partitions, it can get better and better approximations to the right answer.

Refinements

Now that we have stated that it is possible to use the definition directly to compute definite integrals $\int_a^b f(x)\,dx$ numerically (approximately), why do we need to say more?

Well one question is about efficiency. The method might be refined to be more efficient and maybe also to be able to say how approximate the answer is that we get. There are questions like rounding errors that can build up so that longer calculations can become less accurate if not done right.

But there are also cases where one wants to estimate an integral based on sample values of the integrand obtained from an experiment or measurement. In those cases, it may not be possible to keep gathering more data and it may be very desirable to use a method that produces the most accurate answer possible.

There are two basic methods of evaluating definite integrals $\int_a^b f(x)\,dx$ numerically, based on valued of the integrand at $n+1$ evenly spaced values of x.

If the spacing is $h = (b-1)/n$, then we suppose known (from measurements or by calculating $f(x)$) the values of $y = f(x)$ for

$$x = a, x = a + h, x = a + 2h, \ldots, x = a + nh = b$$

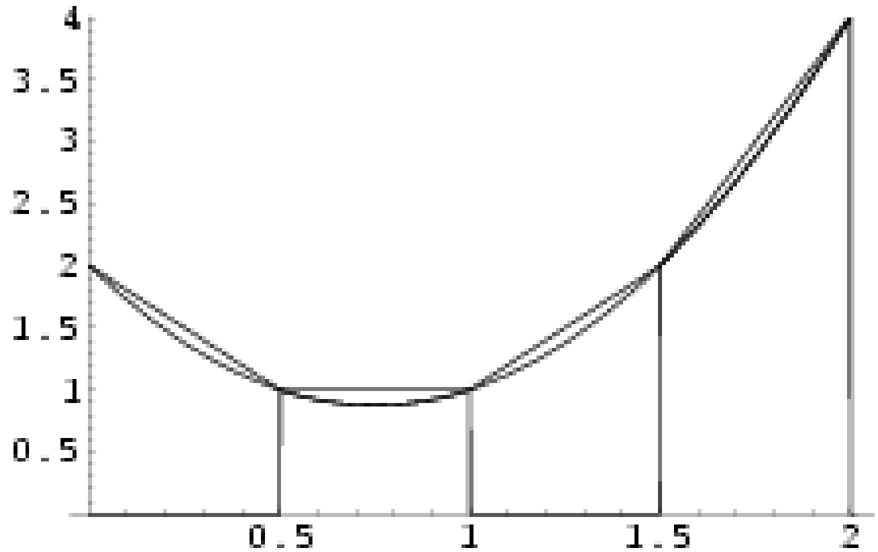
We can write $x_j = a + jh$ for $j = 0, 1, 2, \ldots, n$ and $y_j = f(x_j)$.

3.9.1 Trapezoidal rule formula

(Anton §7.7)

$$\int_a^b f(x)\,dx \cong \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

Here is a picture for $a = 0$, $b = 2$ and $n = 4$ (a very small n). The formula calculates the integral of the function with graph made of bits of straight lines linking the points (x_j, y_j) on the graph $y = f(x)$. It should be a better estimate of the integral that squaring off the vertical strips as one does for Riemann sums.



3.9.2 Simpson's rule formula

(Anton §7.7) Here the notation is as before but n has to be *even*.

$$\int_a^b f(x)\,dx \cong h\left(\frac{1}{3}y_0 + \frac{4}{3}y_1 + \frac{2}{3}y_2 + \frac{4}{3}y_3 + \cdots + \frac{2}{3}y_{n-2} + \frac{4}{3}y_{n-1} + \frac{1}{3}y_n\right)$$

The idea behind this is to replace the actual graph $y = f(x)$ by a graph made up of bits of quadratic graphs. The first quadratic graph goes through (x_0, y_0), (x_1, y_1) and (x_2, y_2). The next starts at (x_2, y_2) and goes through (x_3, y_3) and ends at (x_4, y_4) and so on. Just as there is one line through 2 points, there is one quadratic through 3 points (with different x coordinates) and the formula comes from working out what that graph is and then its integral.

There are theorems (given in Anton) explaining theoretical worst-case error estimates from using the Trapezoidal rule or Simpson's rule. More or less, Simpson's rule is likely to be more accurate than the Trapezoidal rule, or to give a guaranteed accuracy with smaller n. Somewhat more accurately, the worst possible error in the Trapezoidal rule is proportional to h^2 and in Simpson's rule proportional to h^4. If h is fairly small (which means n big since $h = (b-a)/n$), h^4 will be a lot smaller than h^2. However the constants of proportionality can be different, so that the added accuracy in Simpson's rule might not kick in for modest values of n you use.

So if you want to calculate $\ln(2.0) = \int_1^2 (1/x)\,dx$ and you want to be sure that the answer you get is good to (say) 4 decimal places, then you can use these theorem to find an n so that the Trapezoidal rule (or Simpson's rule) with that n will surely be that accurate.

This idea of a worst-case analysis may be unduly cautious in some cases, and in practice the methods can often be used without being sure of the errors. In a case where you only have data, say $11 = 10 + 1$ equally spaced values of the integrand, these worst case results won't be usable because they depend on knowing $f(x)$ and its derivatives. The methods can still be used as estimates, but the errors will need to be managed in some other way.

Richard M. Timoney February 17, 2016